Chapter 1: Further Trancendental Functions

Inverse trigonometric, hyperbolic, inverse hyperbolic

1.1 Inverse Trigonometric Functions

Trigonometric functions are periodic hence they are not one-to one. However, if we restrict the domain to a chosen interval, then the restricted function is one-to-one and invertible.

Definition 1.1 (*The inverse trigonometric functions*)

i) $y = \sin^{-1} x$ if and only if $x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ The function $\sin^{-1} x$ is sometimes written as arcsin *x*. It is necessary that $-1 \leq x \leq 1$.

ii) $y = \cos^{-1} x$ if and only if $x = \cos y$ and $0 \leq y \leq \pi$ The function $\cos^{-1} x$ is sometimes written as arccos *x*. It is necessary that $-1 \leq x \leq 1$.

(iii) $y = \tan^{-1} x$ if and only if $x = \tan y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ The function $\tan^{-1} x$ is sometimes written as arctan *x*. Here *x* may be any value.

- $y = \sin^{-1} x$ is a real number such that $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and sin *y* = *x*. Depending on the context, $\sin^{-1} x$ might also be an angle, a measure of an angle in degrees or radians, or the length of an arc of the unit circle. Likewise for $\cos^{-1} x$, and $\tan^{-1} x$.

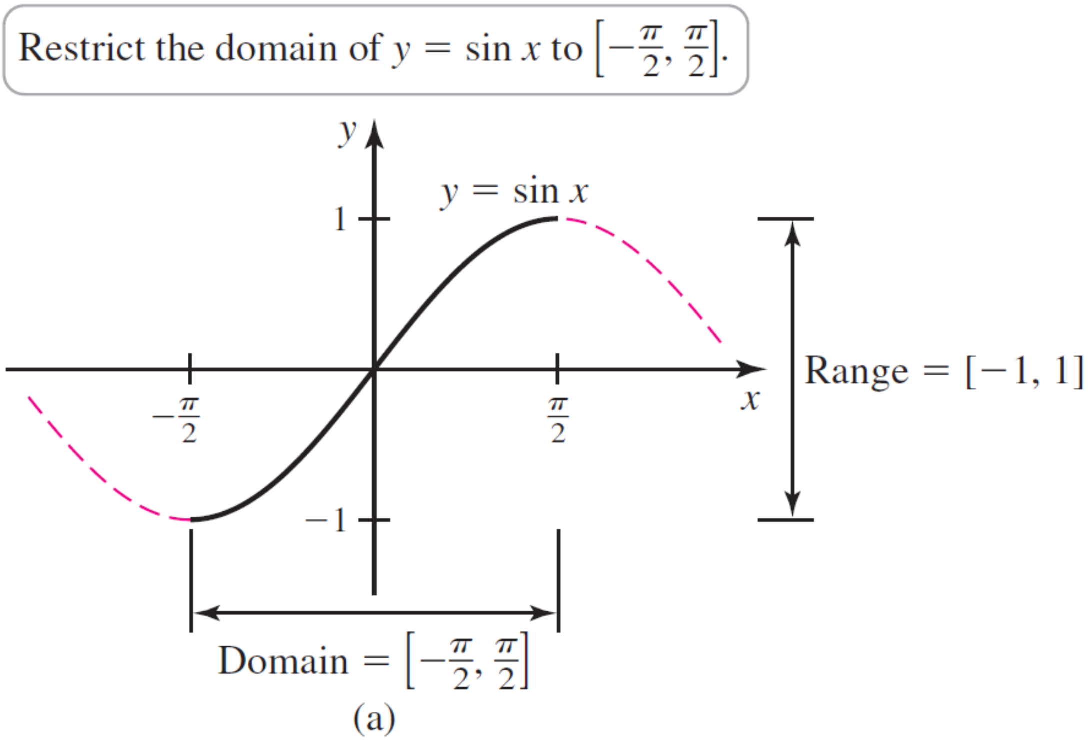
Table of Inverse Trigonometric Functions

Functions	Domain	Range	Quadrants
$y=\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	I & IV
$y=\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	I & II
$y=\tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	I & IV
$y=\csc^{-1} x$	$\lvert x\rvert \geq 1$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$	I & IV
$y=\sec^{-1} x$	$\lvert x\rvert \geq 1$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$	I & II
$y=\cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$	I & II

- It is easier to remember the restrictions on the domain and range if you do so in terms of quadrants.
- $\sin^{-1} x \neq \frac{1}{\sin x}$ whereas $(\sin x)^{-1} = \frac{1}{\sin x}$.

1.11 Graphs of Inverse Trigonometric Functions

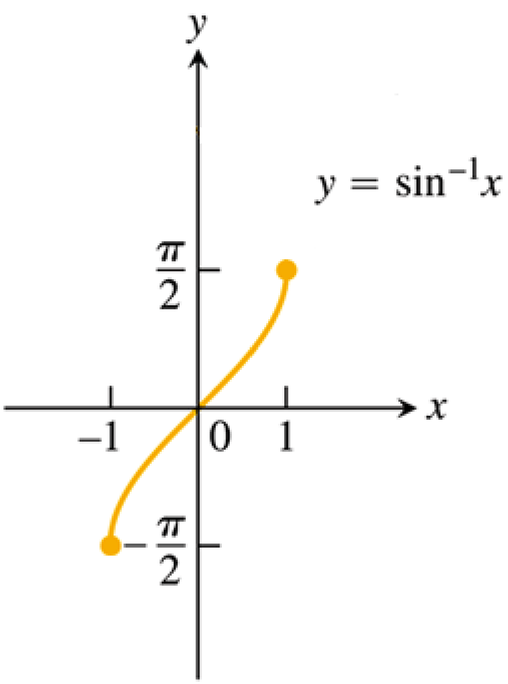
The graphs are reflections about the line $y = x$.

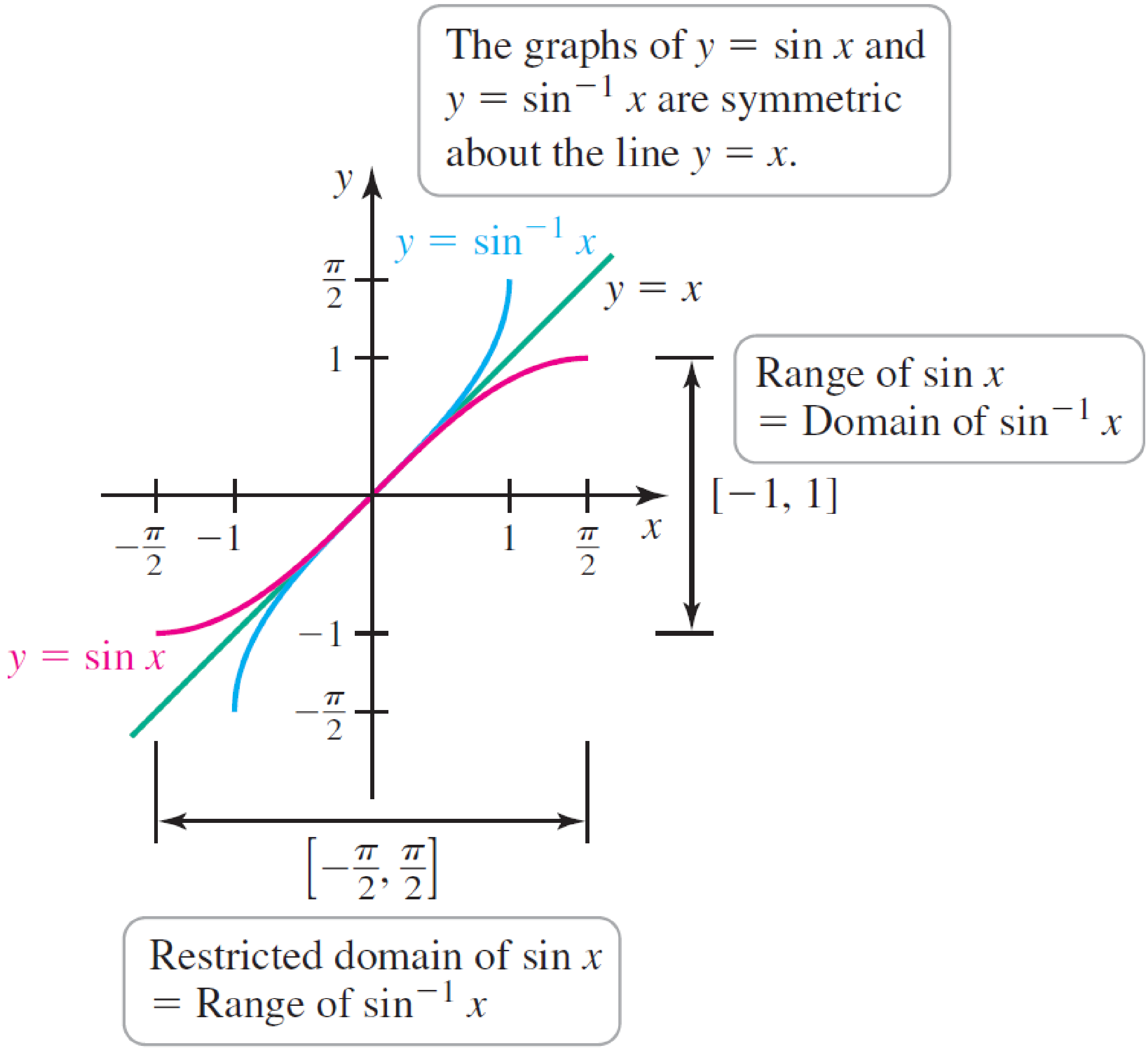


$y = \sin^{-1} x$

Domain:

Range:

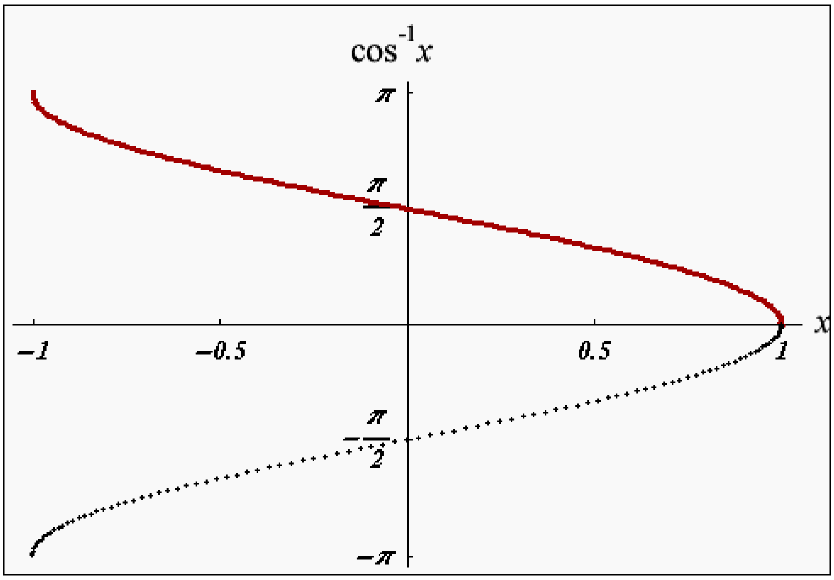


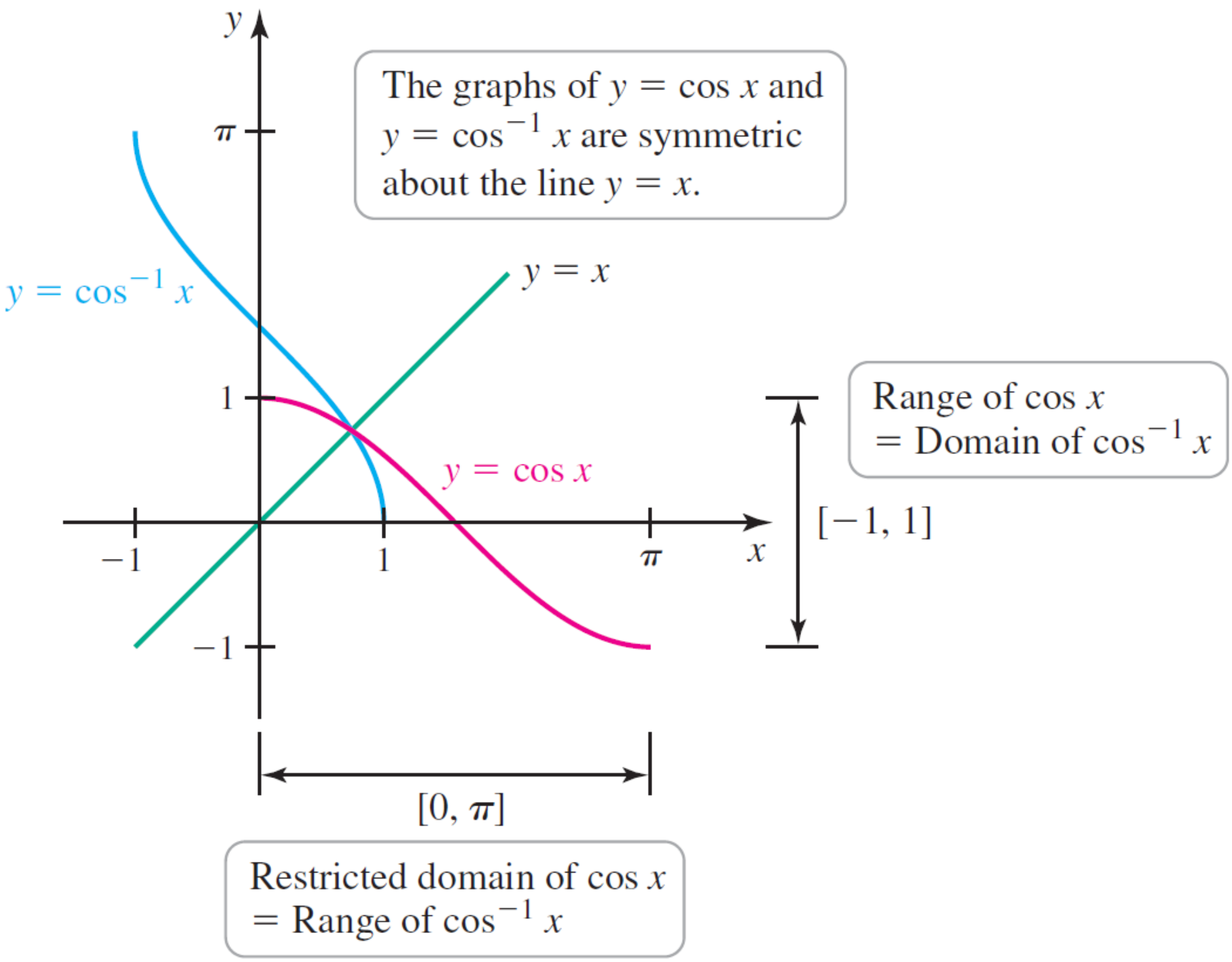

The graphs of $y = \sin x$ and $y = \sin^{-1} x$ are symmetric about the line $y = x$.
$y = \sin^{-1} x$
$y = x$
$y = \sin x$
Range of $\sin x$ = Domain of $\sin^{-1} x$
$[-1, 1]$
$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
Restricted domain of $\sin x$ = Range of $\sin^{-1} x$

$y = \cos^{-1} x$

Domain:

Range:

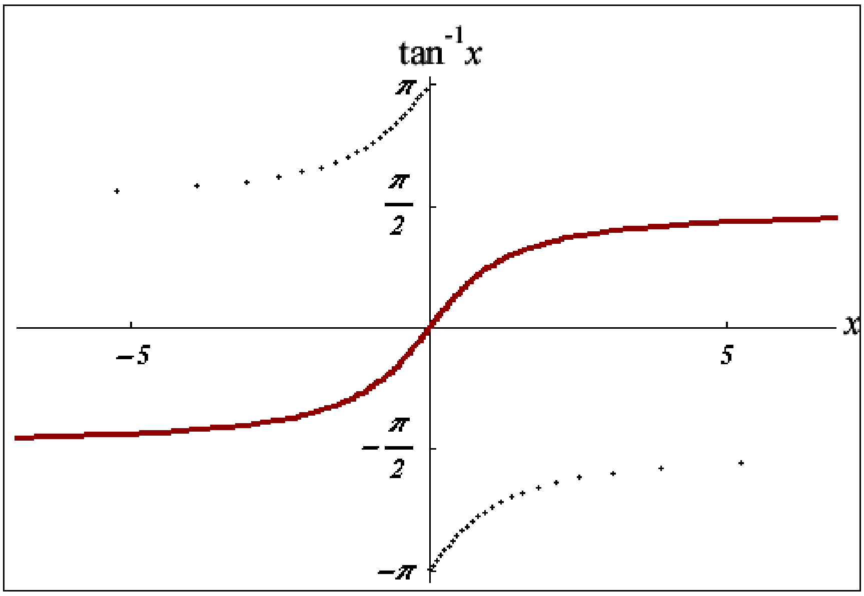




$y = \tan^{-1} x$

Domain:

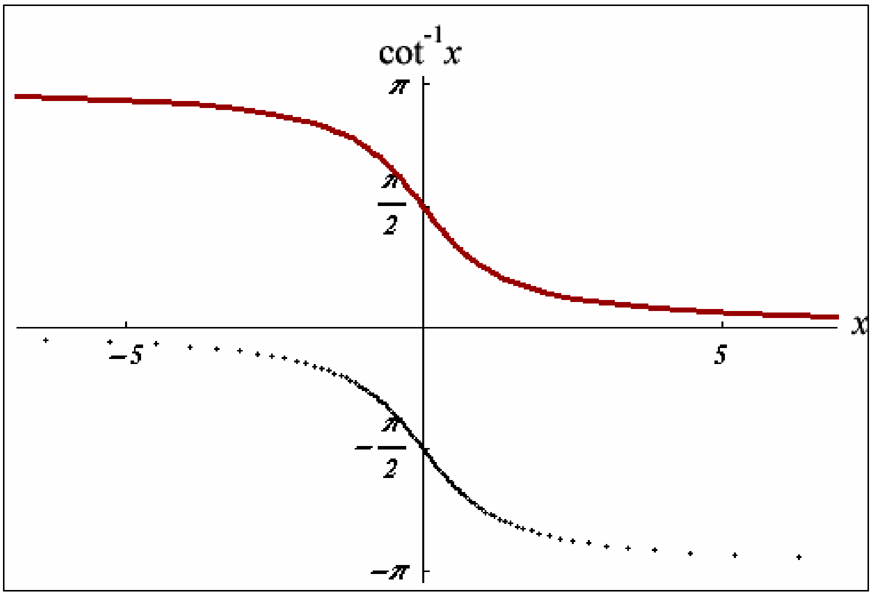
Range:



$y = \cot^{-1} x$

Domain:

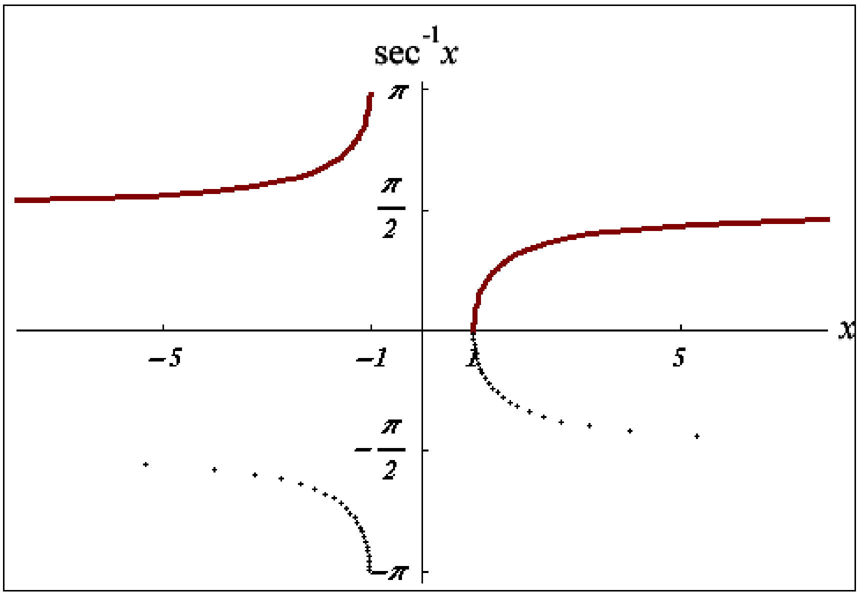
Range:



$y = \sec^{-1} x$

Domain:

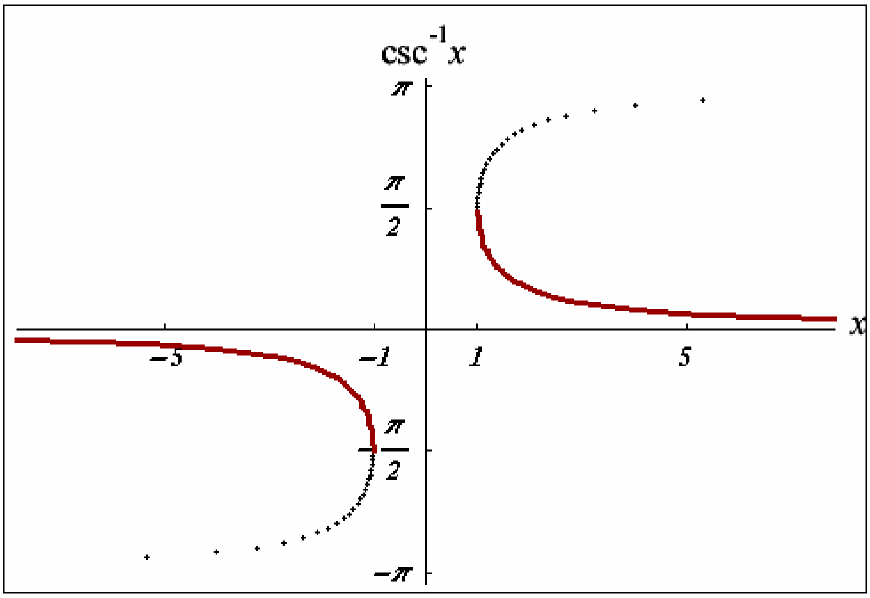
Range:



$y = \csc^{-1} x$

Domain:

Range:



1.12 Evaluating Inverse Trigonometric Functions

Special Values:

x	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$
0	0	1	0
1	$\frac{\pi}{2}$	0	$\frac{\pi}{4}$
$\frac{1}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	
$\frac{1}{\sqrt{2}}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{6}$	
-1	$-\frac{\pi}{2}$	π	
$-\frac{1}{2}$	$-\frac{\pi}{6}$	$\frac{2\pi}{3}$	
$-\frac{1}{\sqrt{2}}$	$-\frac{\pi}{4}$	$\frac{3\pi}{4}$	
$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$	$\frac{7\pi}{6}$	

Example 1.0 Evaluate the given functions.

(i) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

(ii) $\sin^{-1} 0.21$

(iii) $\cos^{-1} 0$

(iv) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

1.13 Inverse Trigonometric Identities

The definition of the inverse functions yields several formulas.

Inversion formulas

$\sin(\sin^{-1} x) = x$	for $-1 \le x \le 1$
$\sin^{-1}(\sin y) = y$	for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$\tan(\tan^{-1} x) = x$	for all x
$\tan^{-1}(\tan y) = y$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

- These formulas are valid only on the specified domain

Reciprocal Identities

$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$	for $\lvert x \rvert \ge 1$
$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$	for $\lvert x \rvert \ge 1$
$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$	for all x

Basic Relation

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$	for $0 \le x \le 1$
$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$	for $0 \le x \le 1$
$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$	$0 \le x \le 1$

Negative Argument Formulas

$\sin^{-1}(-x) = -\sin^{-1} x$	$\sec^{-1}(-x) = \pi - \sec^{-1} x$
$\tan^{-1}(-x) = -\tan^{-1} x$	$\csc^{-1}(-x) = -\csc^{-1} x$
$\cos^{-1}(-x) = \pi - \cos^{-1} x$	$\cot^{-1}(-x) = \pi - \cot^{-1} x$

Example 1.2 Evaluate the given functions.

(i) $\sin(\sin^{-1} 0.5)$ (ii) $\sin(\sin^{-1} 2)$

(iii) $\sin^{-1}(\sin 0.5)$ (iv) $\sin^{-1}(\sin 2)$

Example 1.3 Evaluate the given functions.

(i) $\operatorname{arcsec}(-2)$ (ii) $\csc^{-1}(\sqrt{2})$

(iii) $\cot^{-1} \ -\frac{1}{\sqrt{3}}$

Example 1.4

For $-1 \le x \le 1$, show that

(i) $\sin^{-1}(-x) = -\sin^{-1} x$

(ii) $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$

1.2 Hyperbolic Function

We introduce a new group of mathematical functions, based on the functions

$$e^x \text{ and } e^{-x}$$

Their properties resemble, very closely, those of the standard trigonometric functions.

Just as trigonometric functions can be related to the geometry of a circle, the new functions can be related to the geometry of a hyperbola.

We use the expression of e^x as a series of powers of x as a foundation of the definitions we are going to use.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

If we replace x by $-x$, we get

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$$

The combination of e^x and e^{-x} form what is defined as the hyperbolic function. For instance,

$$\frac{e^x - e^{-x}}{2} = \textit{hyperbolic sine of } x$$

Definition 1.2 (*Hyperbolic functions*)

(i) **Hyperbolic Sine**, pronounced **"shine"**.

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

(ii) **Hyperbolic Cosine,** pronounced **"cosh"**.

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

(iii) **Hyperbolic Tangent,** pronounced **"than"**.

$$\tanh x = \frac{\sinh x}{\cosh x}$$

In terms of exponentials, it is easily shown that

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \equiv \frac{e^{2x} - 1}{e^{2x} + 1}$$

(iv) **Hyperbolic Secant**, pronounced **"shek"**

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

(v) **Hyperbolic Cosecant**, pronounced **"coshek"**

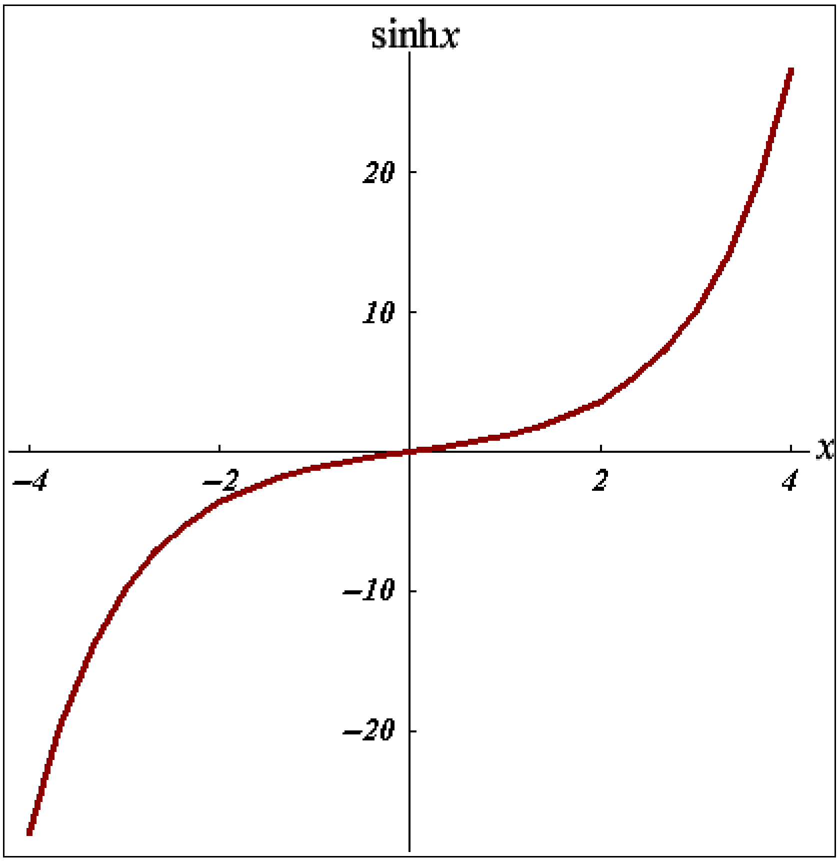
$$\operatorname{cosech} x = \frac{1}{\sinh x}$$

(vi) **Hyperbolic Cotangent**, pronounced **"coth"**

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$$

1.21 Graphs of Hyperbolic Functions

Since the hyperbolic functions depend on the values of e^x and e^{-x}, its graphs is a combination of the exponential graphs.



From the graph, we see

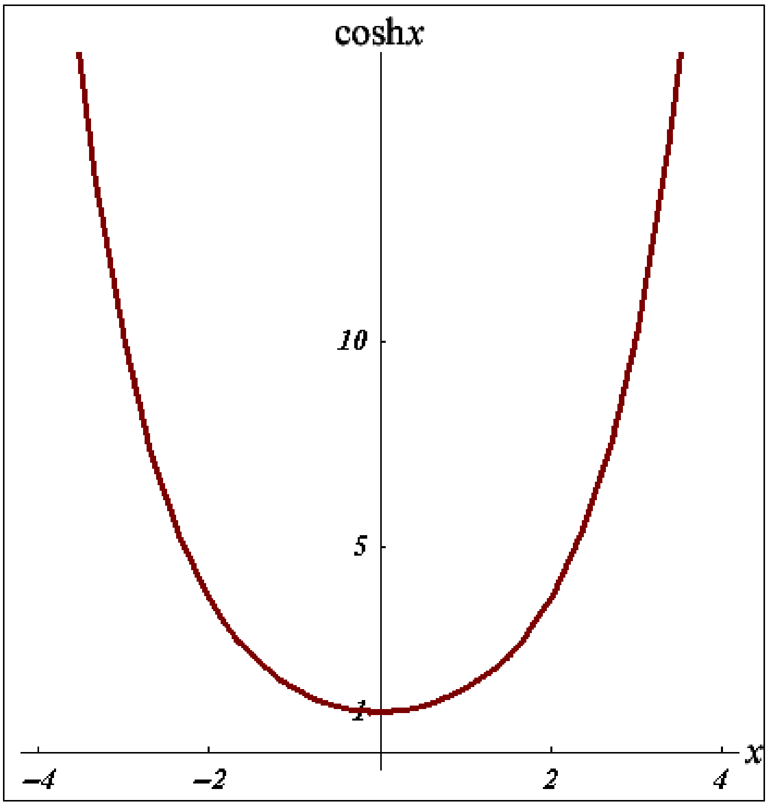
(i) sinh 0 = 0.

(ii) The domain is all real numbers

(iii) The curve is symmetrical about the origin, i.e.

$$\sinh(-x) = -\sinh x$$

(iv) It is an increasing one-to-one function.



We see from the graph of $y = \cosh x$ that:

(i) $\cosh 0 = 1$

(ii) The domain is all real numbers.

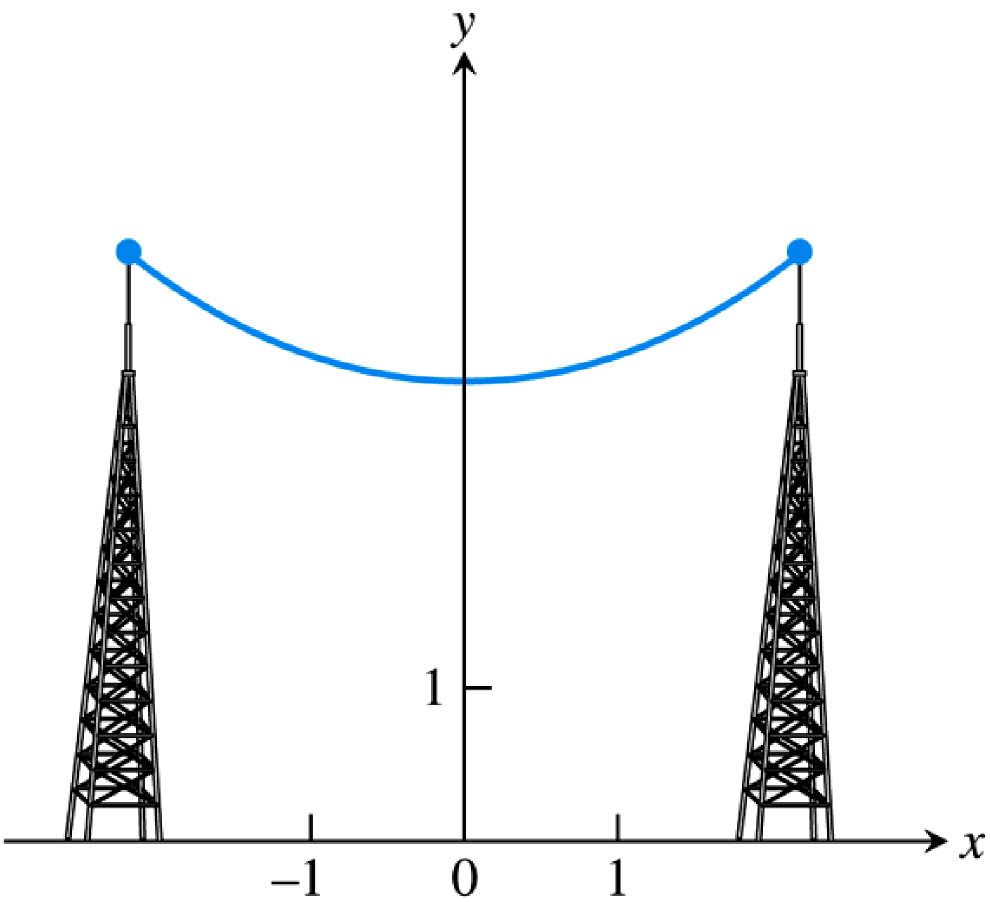
(iii) The value of $\cosh x$ is never less than 1.

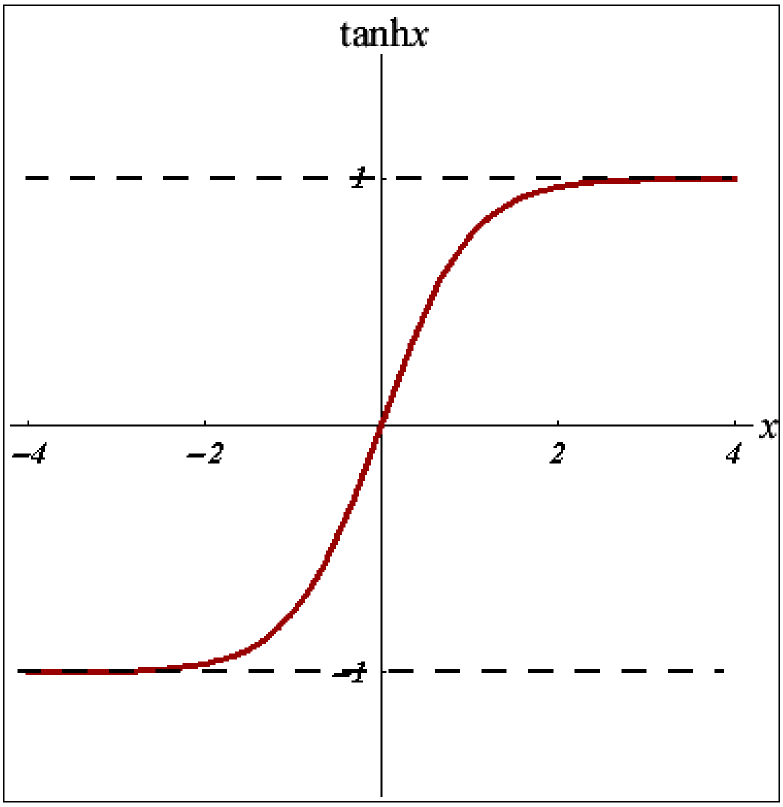
(iv) The curve is symmetrical about the y-axis, i.e.

$$\cosh(-x) = \cosh x$$

(v) For any given value of $\cosh x$, there are two values of x.

Example: Graph of a catenary or hanging cable (the Latin world catena means “chain”)





We see

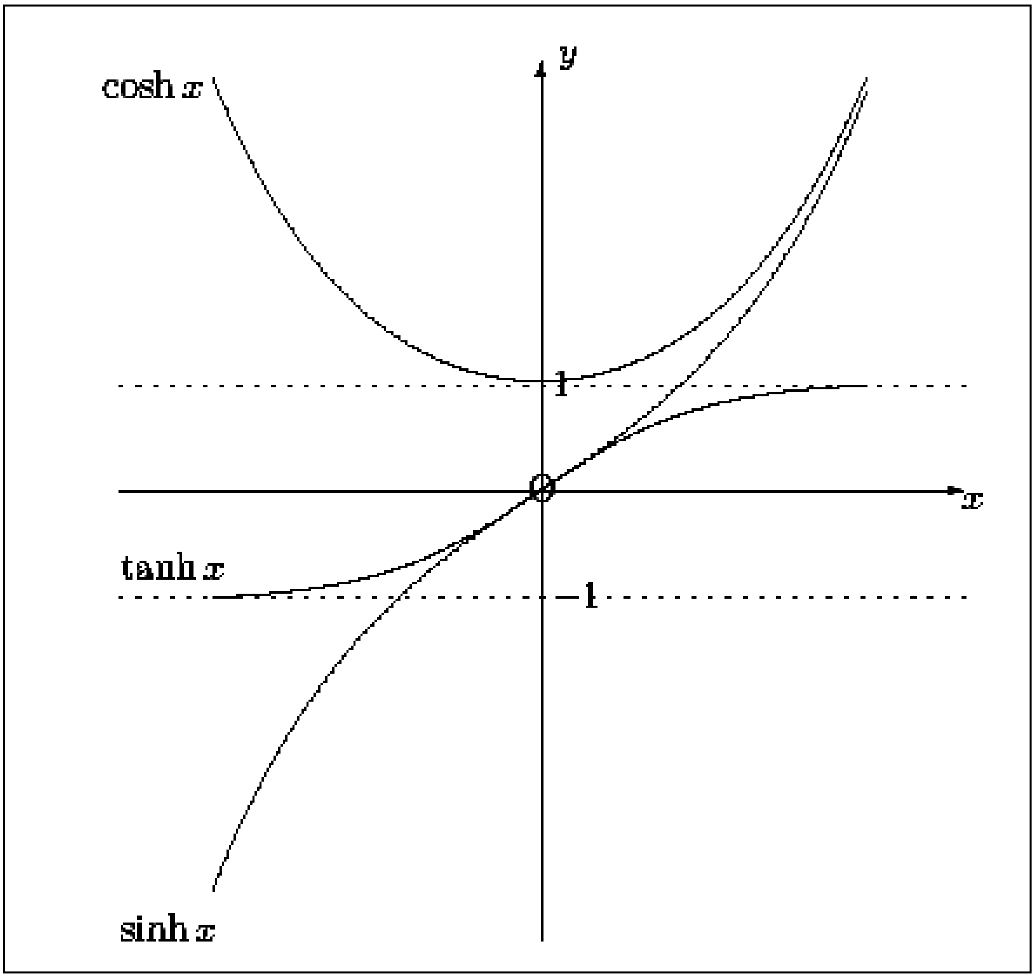
(i) tanh 0 = 0

(ii) tanh x always lies between y = –1 and y = 1.

(iii) tanh (–x) = –tanh x

(iv) It has horizontal asymptotes $y = \pm 1$.

Below are all three graphs on one diagram:



- The graph of cosh *x* exists only for *y* greater than or equal to 1.
- The graph of tanh *x* exists only for *y* lying between –1 and +1.
- The graph of sinh *x* covers the whole range of *x* and *y* values from $-\infty$ to $+\infty$.

1.22 Hyperbolic Identities

For every identity obeyed by trigonometric functions, there is a corresponding identity obeyed by hyperbolic functions.

ILLUSTRATIONS

1.

$$e^x \equiv \cosh x + \sinh x.$$

Proof

$$\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \equiv e^x.$$

2.

$$e^{-x} \equiv \cosh x - \sinh x.$$

Proof

$$\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} \equiv e^{-x}.$$

3.

$$\cosh^2 x - \sinh^2 x \equiv 1$$

Proof

Multiply together the results of the two previous two illustrations;

$$e^x \cdot e^{-x} = 1$$

$$(\cosh x + \sinh x)(\cosh x - \sinh x) \equiv \cosh^2 x - \sinh^2 x$$

Notes:

(i) Dividing throughout by $\cosh^2 x$ gives the identity,

$$1 - \tanh^2 x \equiv \mathrm{sech}^2 x.$$

(ii) Dividing throughout by $\sinh^2 x$ gives the identity,

$$\coth^2 x - 1 \equiv \mathrm{cosech}^2 x.$$

4.

$$\sinh(x + y) \equiv \sinh x \cosh y + \cosh x \sinh y.$$

5. $$\cosh(x+y) \equiv \cosh x \cosh y + \sinh x \sinh y$$

6. $$\tanh(x+y) \equiv \frac{\tanh x + \tanh y}{1 - \tanh x \tanh y}$$

Trig. Identities	**Hyperbolic Identities**
$\sec\theta \equiv \dfrac{1}{\cos\theta}$ $\operatorname{cosec}\theta \equiv \dfrac{1}{\sin\theta}$ $\cot\theta \equiv \dfrac{1}{\tan\theta}$	$\operatorname{sech}\theta = \dfrac{1}{\cosh\theta}$ $\operatorname{cosech}\theta = \dfrac{1}{\sinh\theta}$ $\coth\theta = \dfrac{1}{\tanh\theta}$
$\cos^2\theta + \sin^2\theta \equiv 1$ $1 + \tan^2\theta \equiv \sec^2\theta$ $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$	$\cosh^2\theta - \sinh^2\theta \equiv 1$ $1 - \tanh^2\theta \equiv \operatorname{sech}^2\theta$ $\coth^2\theta - 1 \equiv \operatorname{cosech}^2\theta$
$\sin 2A \equiv 2\sin A\cos A$ $\cos 2A \equiv \cos^2 A - \sin^2 A$ $\equiv 1 - 2\sin^2 A$ $\equiv 2\cos^2 A - 1$	$\sinh 2A \equiv 2\sinh A\cosh A$ $\cosh 2A \equiv \cosh^2 A + \sinh^2 A$ $\equiv 1 + 2\sinh^2 A$ $\equiv 2\cosh^2 A - 1$

- Some of the hyperbolic identities follow exactly the trig. identities; others have a difference in sign.
- The change of sign occurs whenever $\sin^2\theta$ in the trig. results is converted into $\sinh^2\theta$ to form the corresponding hyperbolic identities.

1.23 Evaluation of Hyperbolic Functions

The value of the hyperbolic functions can be calculated using:

(i) calculators

(ii) hyperbolic identities

Example 1.5

(a) Find sinh 4 and cosh (ln 2) correct to 4 d.p.

(b) Express $\sinh(2\ln x)$ in exponential form.

Example 1.6

Given that $\sinh x = \frac{3}{5}$. Using the appropriate identity, find the value of cosh *x* and cosh 2*x*.

1.3 Inverse Hyperbolic Function

The three basic inverse hyperbolic functions are $\cosh^{-1}x$, $\sinh^{-1}x$ and $\tanh^{-1}x$.

Definition 1.3 (*Inverse Hyperbolic Function*)

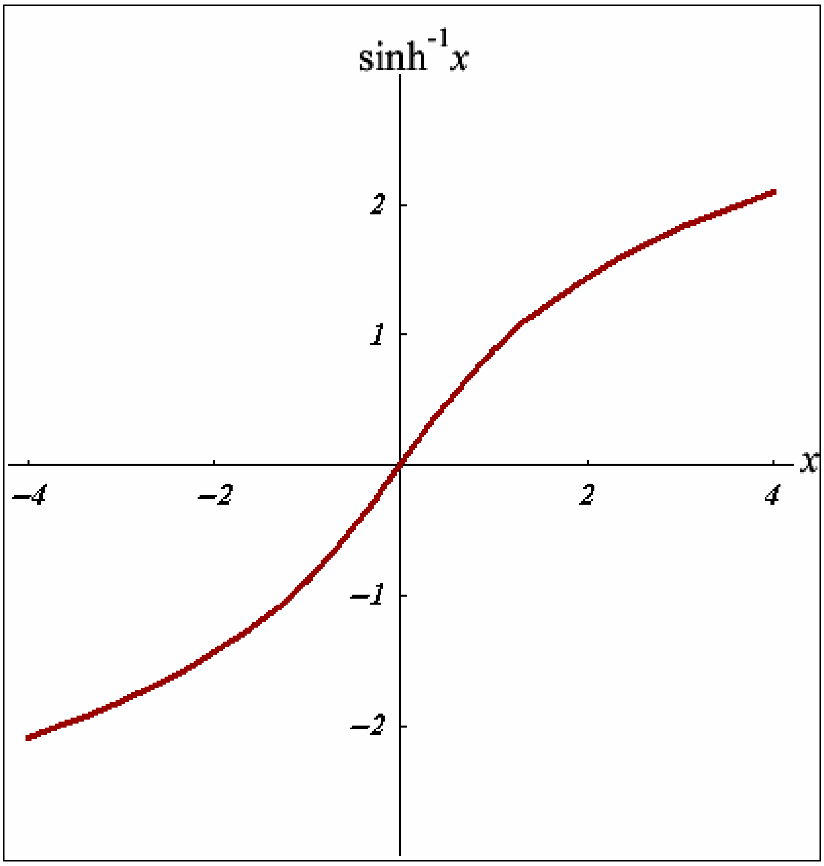
$y=\sinh^{-1}x \;\Leftrightarrow\; x=\sinh x$ for all x and $y\in\mathbb{R}$

$y=\cosh^{-1}x \;\Leftrightarrow\; x=\cosh x$ for $x\geq 1$ and $y\geq 0$

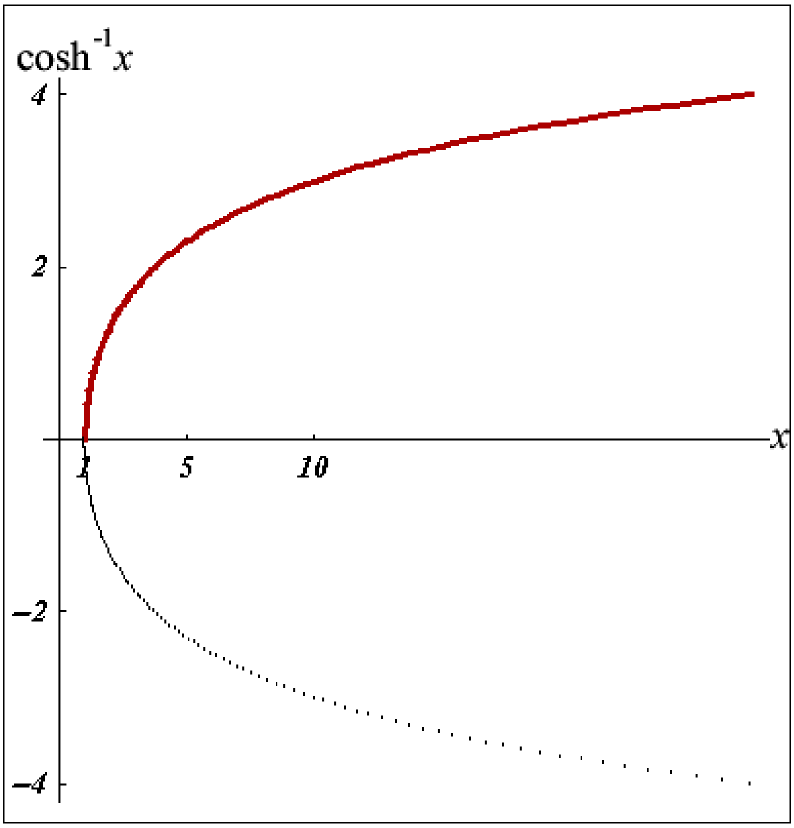
$y=\tanh^{-1}x \;\Leftrightarrow\; x=\tanh x$ for $-1\leq x\leq 1$, $y\in\mathbb{R}$

1.31 Graphs of Inverse Hyperbolic Functions

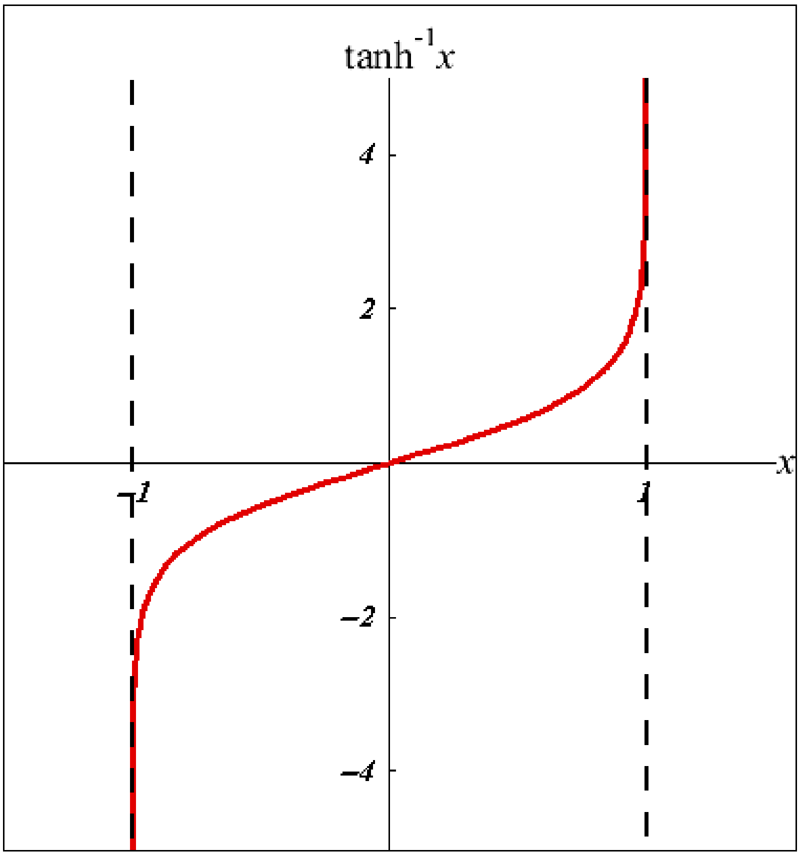
$y=\sinh^{-1}x$
domain:
range:



$y = \cosh^{-1} x$
domain:
range:



$y = \tanh^{-1} x$
domain:
range:



1.32 Log Form of The Inverse Hyperbolic Function

It may be shown that

(a)

$$\text{Cosh}^{-1}x = \pm\ln(x + \sqrt{x^2 - 1}).$$

(b)

$$\text{Sinh}^{-1}x = \ln(x + \sqrt{x^2 + 1}).$$

(c)

$$\text{Tanh}^{-1}x = \frac{1}{2}\ln\frac{1 + x}{1 - x}.$$

Inverse Hyperbolic Cosine

If we let $y = \text{Cosh}^{-1}x$, then

$$x = \cosh y = \frac{e^y + e^{-y}}{2}.$$

Hence,

$$2x = e^y + e^{-y}.$$

On rearrangement,

$$(e^y)^2 - 2xe^y + 1 = 0.$$

Hence,

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}.$$

$\therefore\; e^y = x + \sqrt{x^2 - 1}$ and $e^y = x - \sqrt{x^2 - 1}$.

Both results are positive since $\sqrt{x^2 - 1} < x$. However,

$$\frac{1}{x + \sqrt{x^2 - 1}} = x - \sqrt{x^2 - 1}.$$

That is $e^y = x + \sqrt{x^2 - 1}$ or $\left\{x + \sqrt{x^2 - 1}\right\}^{-1}$

Taking natural logarithms,

$$y = \pm \ln\left(x + \sqrt{x^2 - 1}\right)$$

In the same way, we can find the expression for $\sinh^{-1} x$ and $\tanh^{-1} x$ in logaritmic form.

Example 1.7 Evaluate

(i) $\sinh^{-1}(0.5)$

(ii) $\cosh^{-1}(0.5)$

(iii) $\tanh^{-1}(-0.6)$

***Insert Application Example

Exercises

1. Write down the principal values of the following:

(a) $\sin^{-1} 1$ (b) $\sin^{-1}\left(-\frac{1}{2}\right)$

(c) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (d) $\tan^{-1} 5$

(e) $\tan^{-1}(-\sqrt{3})$ (f) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

(g) $\sec^{-1}(-\sqrt{2})$ (h) $\csc^{-1}(-\sqrt{2})$

(i) $\cot^{-1}(-\sqrt{3})$

2. Solve the following equations for x in the interval $0^\circ \le x \le 360^\circ$.

(a) $\tan x = 2.46$

(b) $\cos x = 0.241$

(c) $\sin x = -0.786$

(d) $\tan x = -1.42$

(e) $\cos x = -0.3478$

(f) $\sin x = 0.987$

3. Given that $A = \sin^{-1}\left(\frac{1}{4}\right)$. Find $\sin 2A$.

4. Find the values of the following functions:

(a) $\cos\left(\sin^{-1}\frac{1}{2}\right)$ (b) $\sin\left(\cos^{-1}\frac{1}{\sqrt{2}}\right)$

(c) $\cot\left(\tan^{-1}\frac{1}{3}\right)$ (d) $\tan\left(\sin^{-1}\frac{1}{3}\right)$

(e) $\cos\left(\sin^{-1}\frac{1}{5} + 2\cos^{-1}\frac{1}{5}\right)$

5. Simplify each expression:

(a) $\sin(2\tan^{-1} x)$ (b) $\cos(2\sin^{-1} x)$

6. Use reference triangles to justify the following identities.

(a) $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$ for all x.

(b) $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$ for all $|x| \geq 1$

7. Solve for x the equation $\sqrt{x} = \cos^{-1} 0.317 + \sin^{-1} 0.317$.

8. If $\tanh x = \frac{1}{3}$, find e^{2x} and hence evaluate x.

9. Evaluate (a) $\sinh^{-1} 7$ (b) $\cosh^{-1} 9$ (c) $\tanh^{-1} 0.75$

10. Express cosh $2x$ and sinh $2x$ in terms of exponential and hence solve for x, the equation

$$2\cosh 2x - \sinh 2x = 2$$

11. Obtain a formula which equates $\operatorname{cosech}^{-1} x$ to the natural logarithm of an expression in x, distinguishing between the two cases $x > 0$ and $x < 0$.

12. If $t = \tanh\left(\frac{x}{2}\right)$, prove that

$$\sinh x = \frac{2t}{1-t^2} \text{ and } \cosh x = \frac{1+t^2}{1-t^2}.$$

Hence solve for x, the equation

$$7 \sinh x + 20 \cosh x = 24.$$

13. Prove that $\tanh^{-1}\left\{\frac{x^2-1}{x^2+1}\right\} = \ln x$.

Answers

1. (a) $\frac{\pi}{2}$; (b) $-\frac{\pi}{6}$; (c) $\frac{5\pi}{6}$; (d) 1.373; (e) $-\frac{\pi}{3}$; (f) $\frac{3\pi}{4}$. ; (g) $\frac{3\pi}{4}$; (h) $\frac{\pi}{4}$; (i) $\frac{5\pi}{6}$

2. (a) 67.9° or 247.9°;
(b) 76.1° or 283.9°;
(c) 231.8° or 308.2°;
(d) 125.2° or 305.32°;
(e) 110.4° or 249.6°;
(f) 80.8° or 99.2°

3. $\dfrac{\sqrt{15}}{8}$

4. (a) $\dfrac{\sqrt{3}}{2}$; (b) $\dfrac{\sqrt{2}}{2}$; c) 3; (d) $\dfrac{\sqrt{2}}{4}$; (e) $\dfrac{-2\sqrt{6}}{5}$

5. (a) $\dfrac{2x}{x^2+1}$; (b) $1-2x^2$

7. $\dfrac{\pi^2}{4}$

8. $x = 0.3466$

9. (a) $\ln\left(7+\sqrt{49+1}\right) \approx 2.644$; (b) $\ln\left(9+\sqrt{81-1}\right) \approx 2.887$; (c) 9731

10. $x = 0$ or $\dfrac{1}{2}\ln 3$

11. If $x > 0$, $\operatorname{cosech}^{-1} x = \ln \dfrac{1+\sqrt{1+x^2}}{x}$.

If $x < 0$, $\operatorname{cosech}^{-1} x = \ln \dfrac{1-\sqrt{1+x^2}}{x}$.

12. $x \approx -1.099$ or 0.386