Chapter 6: Vectors

6.1 Basic concepts

6.2 Dot product

- Definition
- Angle between 2 vectors

6.3 Cross product

- Definition
- Area of parallelogram/triangle

6.4 Lines in space

- Parametric and symmetric equation
- Angle between two lines
- Intersection of two lines
- Distance from a point to a line

6.5 Planes in Space

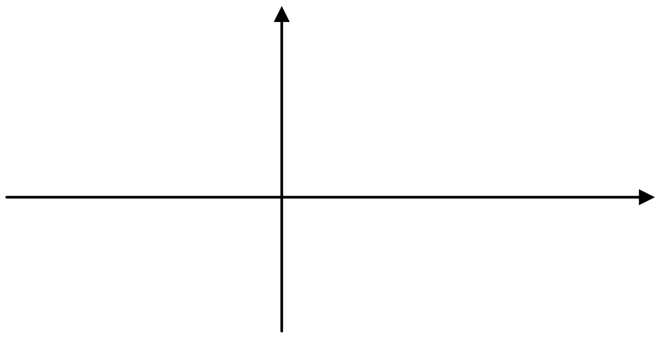
- Equation of a plane
- Intersection of two planes
- Angle between two planes
- Angle between a line and a plane
- Shortest distance
  - from a point to a plane
  - between two parallel planes
  - between two skewed lines

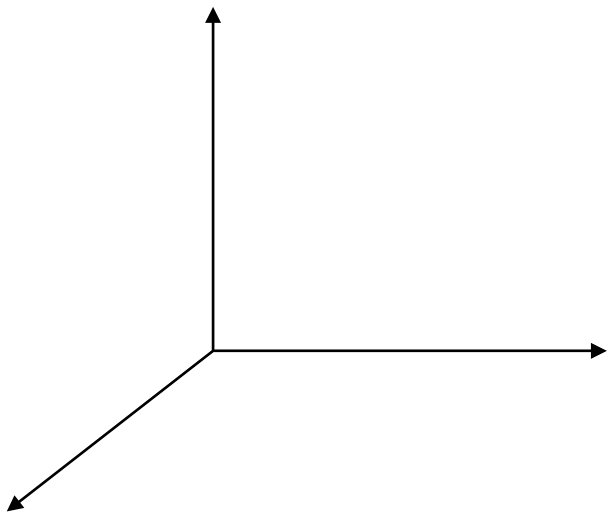
6.1: Basic concepts

Vector: quantity that has both magnitude and direction.
E.g: Force, velocity.

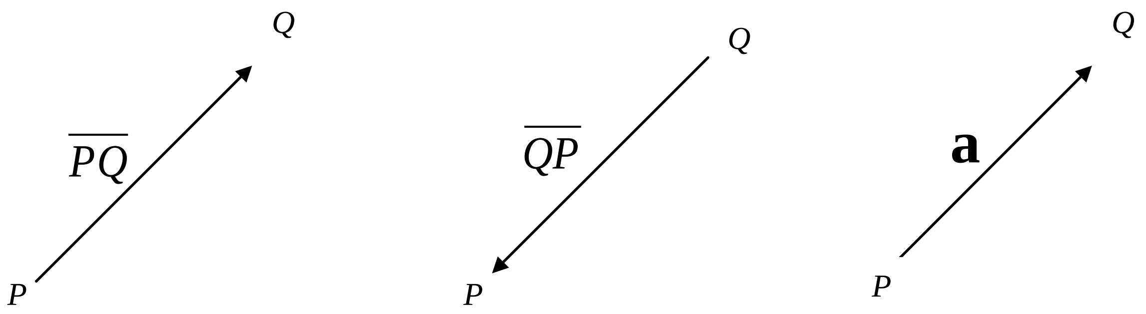
A vector can be represented by a directed line segment where the

i) length of the line represents the magnitude
ii) direction of the line represents the direction





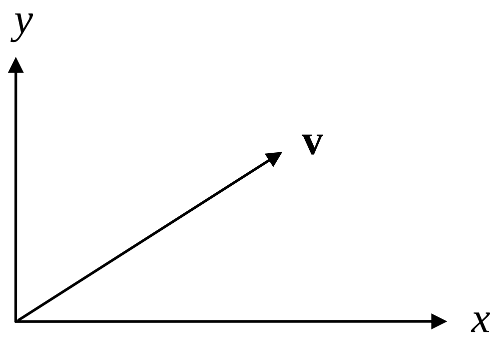
Notation:



Vector components:

$$\bar{v} = a\,\underline{i} + b\,\underline{j}$$

a and *b*: scalar component
i and *j* : direction



In 3D:

$\bar{v} = a\,\underline{i} + b\,\underline{j} + c\underline{k}$ or $\bar{v} = < a, b, c >$

Note that $\bar{v} = < a, b, c > \neq \bar{v} = (a, b, c)$

- The vector $\vec{PQ}$ with initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$ has the standard representation

$$\overline{PQ} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

Or $\overline{PQ} = < x_2 - x_1, y_2 - y_1, z_2 - z_1 >$

Important Formulae

Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ be vectors in 3D space and k is a constant.

1. Magnitude

$$|\mathbf{v}| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$$

2. Unit vector in the direction of **v** is

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle v_1, v_2, v_3 \rangle}{|\mathbf{v}|}$$

3. $\mathbf{v} \pm \mathbf{w} = \langle v_1 \pm w_1, v_2 \pm w_2, v_3 \pm w_3 \rangle$

Example 1:

Given that $\mathbf{a} = \langle 3,1,-2 \rangle$, $\mathbf{b} = \langle -1,6,4 \rangle$. Find

(a) $\mathbf{a} + 3\mathbf{b}$ (b) $|\mathbf{b}|$

(c) a unit vector in the direction of **b**.

Example 2: (Test 1, Sem 1 2006/07)

Given the vectors $\mathbf{u} = 3\underline{i} + \underline{j} - 5\underline{k}$ and $\mathbf{v} = 4\underline{i} - 2\underline{j} + 7\underline{k}$.

a) Find a unit vector in the direction of 2**u** + **v.**

Example 3:

Given two points, P(1,0,1) and Q(3,2,0).

Find a unit vector **u** in the direction of $\overline{PQ}$.

6.2 The Dot Product (The Scalar Product)

The scalar product between two vectors

v = $< v_1, v_2, v_3 >$ and **w** = $< w_1, w_2, w_3 >$ is defined as follows:

in components

$$\mathbf{v} \cdot \mathbf{w} = \langle v_1, v_2, v_3 \rangle \cdot \langle w_1, w_2, w_3 \rangle$$
$$= v_1 w_1 + v_2 w_2 + v_3 w_3$$

geometrically

$$\mathrm{v} \cdot \mathrm{w} = |\mathrm{v}||w| \cos\theta$$

where θ is the angle between **v** and **w.**

Example 1: (Test 1, Sem 1 2006/07)

Given the vectors $\mathbf{u} = 3\underline{i} + \underline{j} - 5\underline{k}$ and $\mathbf{v} = 4\underline{i} - 2\underline{j} + 7\underline{k}$.

a) Find the angle between **u** and **v**.

Example 2: (Final Sem 1, 2005/06)

The coordinates of *A*,*B* and *C* are *A*(1,1,-1), *B*(-1,2,3) and *C*(-2,1,1). Find the angle *ABC*, giving your answer to nearest degree.

Example 3: (Final Sem 2, 2006/07)

Given the vectors .

Find the angle between **a** and **b**.

Example 4: (Test 1, 2005/06)

Given **u** = *m***i** + **j** and **v** = 3**i** + 2**j**. Find the values of *m* if the angle between u and v is **─.**   Ans: 1/5, -5

Theorem 6.1:(Angle between two vectors)

The nature of an angle θ, between two vectors **u** and **v**.

1. θ is an acute angle if and only if $\mathbf{u} \cdot \mathbf{v} > 0$
2. θ is an obtuse angle if and only if $\mathbf{u} \cdot \mathbf{v} < 0$
3. $\theta = 90^\circ$ if and only if $\mathbf{u} \cdot \mathbf{v} = 0$

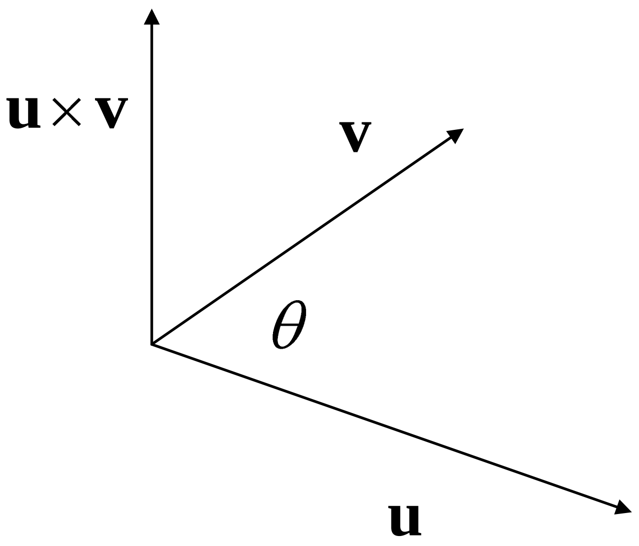
Example:(Final 2004/05)

Given $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \alpha\,\mathbf{j} - 5\mathbf{k}$.

a) Find the value of α if the vectors **a** and **b** are orthogonal.

6.3 The Cross Products (Vector Products)

The cross product (vector product) $\mathbf{u}\times\mathbf{v}$ is a vector perpendicular to **u** and **v** whose direction is determined by the right hand rule and whose length is determined by the lengths of **u** and **v** and the angle between them.



Theorem 6.2 :(cross product)

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathrm{u}\times\mathrm{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

.

Definition 6.1: (Magnitude of Cross Product)

If **u** and **v** are nonzero vectors, and θ ($0 < \theta < \pi$) is the angle between **u** and **v**, then

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta,$$

Theorem 6.3 (Properties of Cross Product)

The cross product obeys the laws

(a) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

(b) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(c) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

(d) $(k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v}) = k(\mathbf{u} \times \mathbf{v})$

(e) **u** // **v** if and only if $\mathrm{u} \times \mathbf{v} = 0$

(f) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

Example 1:

Given that $\mathbf{u} = \langle 3,0,4 \rangle$ and $\mathbf{v} = \langle 1,5,-2 \rangle$, find

(a) $\mathbf{u} \times \mathbf{v}$ (b) $\mathbf{v} \times \mathbf{u}$

Example 2:(Final 2004/05)

Given $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$.

Find a <u>unit vector</u> which is orthogonal to the vectors **a** and **b**.

Example 3: (Final Sem 1, 2005/2006)

Find a unit vector perpendicular to both vectors

Area of parallelogram & triangle

Area of a parallelogram $= |\mathbf{u}||\mathbf{v}|\sin\theta = |\mathbf{u}\times\mathbf{v}|$

Area of triangle = $\frac{1}{2}|\mathbf{u}\times\mathbf{v}|$

Example 1: (Final Sem 2, 2006/2007)

Find an area of a parallelogram bounded by two vectors

Example 2:

Find an area of a triangle that is formed from vectors

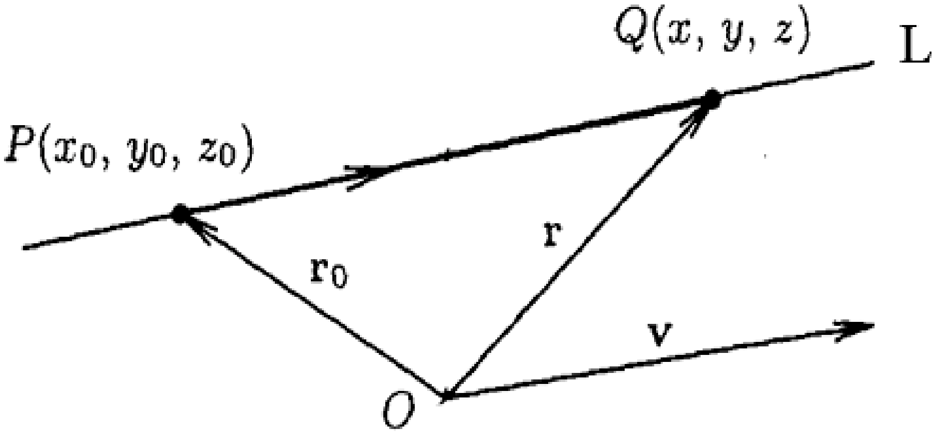
$\mathbf{u} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = -6\mathbf{j} + 5\mathbf{k}$.

Example 3:

Find the area of the triangle having vertices at P (1,3,2), Q(-2,1,3) and R(3,-2,-1). Ans: 11.52sq units.

6.4 Lines in Space

6.4.1 How lines can be defined using vectors?



Suppose **L** is a straight line that passes through $P(x_0, y_0, z_0)$ and is parallel to the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Thus, a point $Q(x, y, z)$ also lies on the line **L** if vectors $\overline{PQ}$ and **v** are parallel, that is:

$$\overline{PQ} = t\,\mathbf{v}$$

Say $\mathbf{r}_0 = \overline{OP}$ and $\mathbf{r} = \overline{OQ}$

$$\therefore \overline{PQ} = \mathbf{r} - \mathbf{r_0}$$

$$\mathbf{r} - \mathbf{r}_0 = t\mathbf{v} \quad \text{or} \quad \mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

In component form,

$$< x, y, z > = < x_0, y_0, z_0 > + t < a, b, c >$$

(equation of line in vector component)

<u>Theorem 1.11(Parametric Equations for a Line)</u>

The line through the point $P(x_0, y_0, z_0)$ and parallel to the nonzero vector $\mathbf{V} = \langle a, b, c \rangle$ has the **parametric equations**

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

Example 1:
Give the parametric equations for the line through the point (6,4,3) and parallel to the vector $\langle 2, 0, -7 \rangle$.

Example 2: (Final 2004/05)

The position vectors of points *A* and *B* are

and . Find the parametric equation of the line *AB*.

<u>Theorem 6.4 (Symmetric Equations for a line)</u>

The line through the point $P(x_0, y_0, z_0)$ and parallel to the nonzero vector $\mathbf{V} = \langle a, b, c \rangle$ has the **symmetrical equations**

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example 1:

Given that the symmetrical equations of a line in space is

$\frac{2x+1}{3} = \frac{3-y}{4} = \frac{z+4}{2}$, find

(a) a point on the line.

(b) a vector that is parallel to the line.

Example 2: (Test 1, 2006/07)

The line l is passing through the points $X(2,0,5)$ and $Y(-3,7,4)$.

Write the equation of l in symmetrical form.

Example 3: (Test 1, 2005/06)

Given a line L: .

Write the equation of L in symmetrical form.

6.4.2 Angle Between Two Lines

Consider two straight lines

$$l_1 : \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

and $$l_2 : \frac{x - x_2}{d} = \frac{y - y_2}{e} = \frac{z - z_2}{f}$$

The line l_1 parallel to the vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and the line l_2 parallel to the vector $\mathbf{v} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$. Since the lines l_1 and l_2 are parallel to the vectors u and v respectively, then the angle, θ between the two lines is given by

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

***Example 1**:*

Find an acute angle between line

$$l_1 = \mathbf{i} + 2\mathbf{j} + t(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

and line

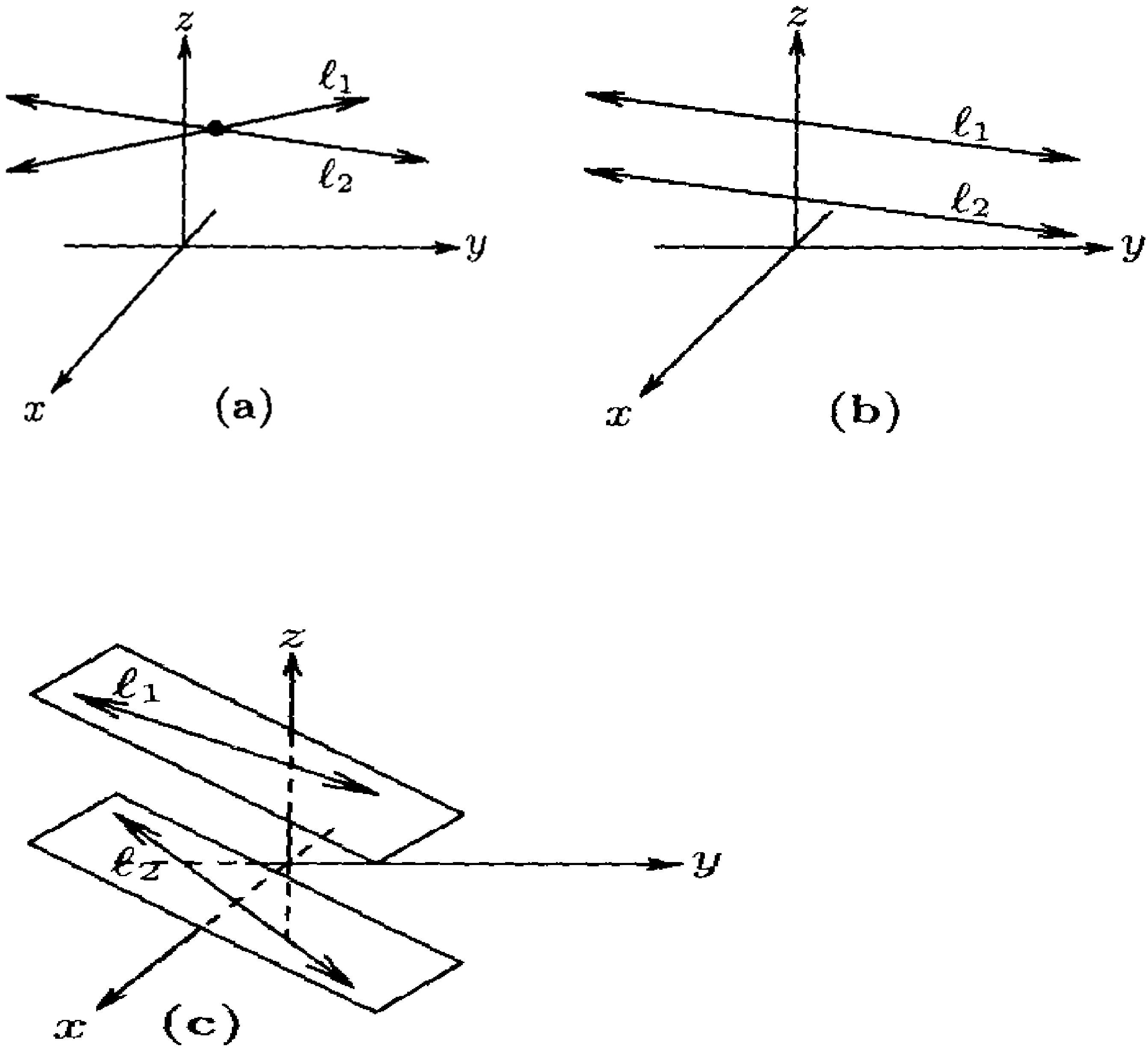
$$l_2 = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + s(3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}).$$

Example 2: (Test 1, 2006/07)

Find the angle between lines l_1 and l_2 which are defined by

6.4.3 Intersection of Two Lines

In three-dimensional coordinates (space), two lines can be in one of the three cases as shown below



a) intersect b) parallel c)skewed

Let l_1 and l_2 are given by:

$$l_1: \; \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad \text{and} \qquad (1)$$

$$l_2: \; \frac{x - x_2}{d} = \frac{y - y_2}{e} = \frac{z - z_2}{f} \qquad (2)$$

From (1), we have $\mathbf{v}_1 = < a, b, c >$

From (2), we have $\mathbf{v}_2 = < d, e, f >$

Two lines are <u>parallel</u> if we can write

$$\mathbf{v}_1 = \lambda \, \mathbf{v}_2$$

The parametric equations of l_1 and l_2 are:

$$l_1: \quad x = x_1 + at$$
$$y = y_1 + bt$$
$$z = z_1 + ct$$

$$l_2: \quad \left. \begin{aligned} x &= x_2 + ds \\ y &= y_2 + es \\ z &= z_2 + fs \end{aligned} \right\} \quad (3)$$

Two lines are intersect if there exist unique values of t and s such that:

$$x_1 + at = x_2 + ds$$
$$y_1 + bt = y_2 + es$$
$$z_1 + ct = z_2 + fs$$

Substitute the value of t and s in (3) to get x, y and z. The point of intersection = (x, y, z)

Two lines are skewed if they are neither parallel nor intersect.

Example 1:

Determine whether l_1 and l_2 are parallel, intersect or skewed.

a) $l_1: \ x = 3 + 3t \ , \ y = 1 - 4t, \ \ z = -4 - 7t$

$l_2: \ x = 2 + 3s \ , \ y = 5 - 4s, \ \ z = 3 - 7s$

b) $l_1: \ \dfrac{x-1}{1} = \dfrac{2-y}{4} = z$

$l_2: \ \dfrac{x-4}{-1} = y - 3 = \dfrac{z+2}{3}$

Solutions:

a) <u>for</u> l_1 :

point on the line, P = (3, 1, - 4)

vector that parallel to line, $\mathbf{v}_1 = < 3, -4, -7 >$

<u>for</u> l_2:

point on the line, Q = (2, 5, 3)

vector that parallel to line, $\mathbf{v}_2 = < 3, -4, -7 >$

$$\mathbf{v}_1 = \lambda \mathbf{v}_2 \qquad ?$$

$$\mathbf{v}_1 = \mathbf{v}_2 \qquad \text{where } \lambda = 1$$

Therefore, lines l_1 and l_2 are parallel.

b) Symmetrical eq's of l_1 and l_2 can be rewrite as:

$$l_1: \ \frac{x-1}{1} = \frac{y-2}{-4} = \frac{z-0}{1}$$

$$l_2: \ \frac{x-4}{-1} = \frac{y-3}{1} = \frac{z-(-2)}{3}$$

Therefore:

<u>for</u> l_1 : P = (1, 2, 0) , $\mathbf{v}_1 = <1, -4, 1>$

<u>for</u> l_2: Q = (4, 3, -2) , $\mathbf{v}_2 = <-1, 1, 3>$

$\mathbf{v}_1 = \lambda \mathbf{v}_2$?

$\mathbf{v}_1 \neq \lambda \mathbf{v}_2 \ \rightarrow$ not parallel.

In parametric eq's:

$l_1: \ x = 1 + t \ , \ y = 2 - 4t \ , \ z = t$

$l_2: \ x = 4 - s \ , \ y = 3 + s \ , \ z = -2 + 3s$

$$1 + t = 4 - s \qquad (1)$$

$$2 - 4t = 3 + s \qquad (2)$$

$$t = -2 + 3s \qquad (3)$$

Solve the simultaneous equations (1), (2), and (3) to get *t* and *s*.

$$s = \frac{5}{4} \quad \text{and} \quad t = \frac{7}{4}$$

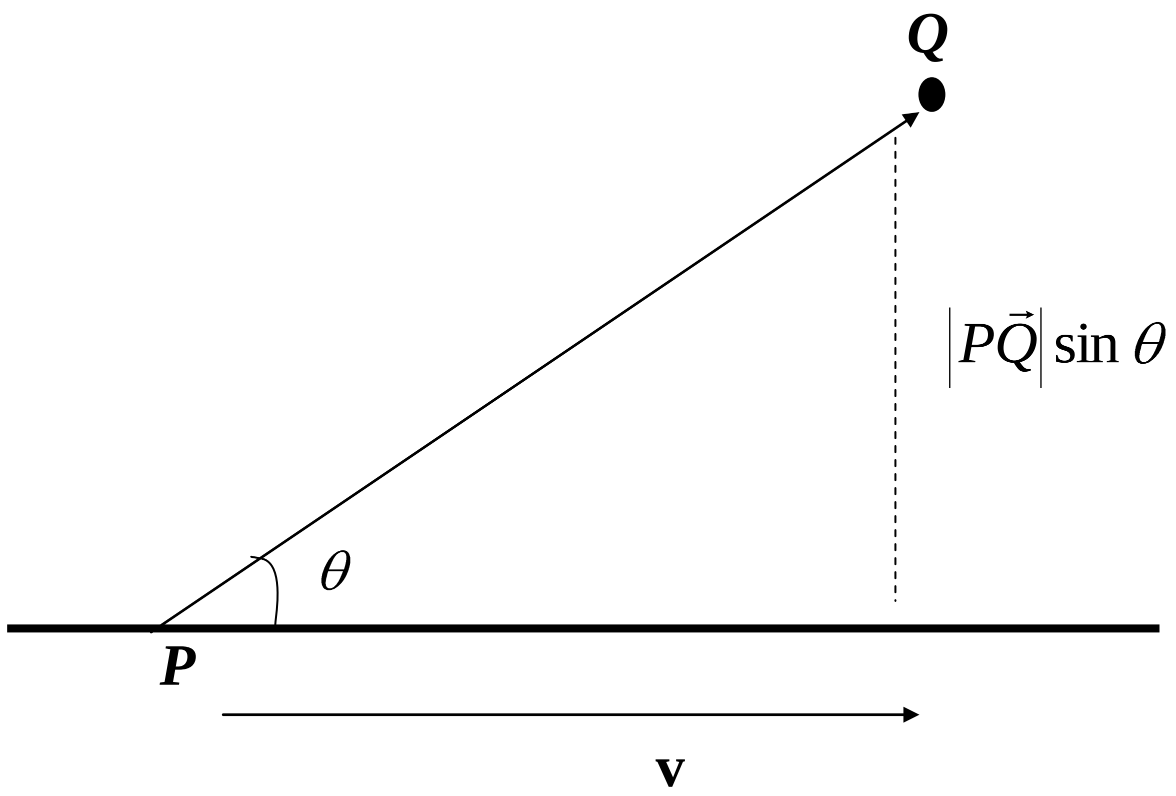
The value of *t* and *s* must satisfy (1), (2), and (3). Clearly they are not satisfying (2) i.e

$$2 - \frac{7}{4} = 3 + \frac{5}{4} \quad ?$$

$$\frac{1}{4} \neq \frac{17}{4}$$

Therefore, lines l_1 and l_2 are not intersect.
This implies the lines are skewed!

6.4.4 Distance From A Point To A Line



Distance from a point Q to a line that passes through point P parallel to vector $\mathbf{v}$ is equal to the length of the component of PQ perpendicular to the line.

$$d = \left|\overline{PQ}\right| \sin\theta$$

$$= \frac{\left|\overline{PQ}\right| \times \mathbf{v}}{\mathbf{v}}$$

Example 1:(Test 1, 2005/06)

Given a line L: .

Find the shortest distance from a point Q(4,1,-2) to the line L.

Example 2:

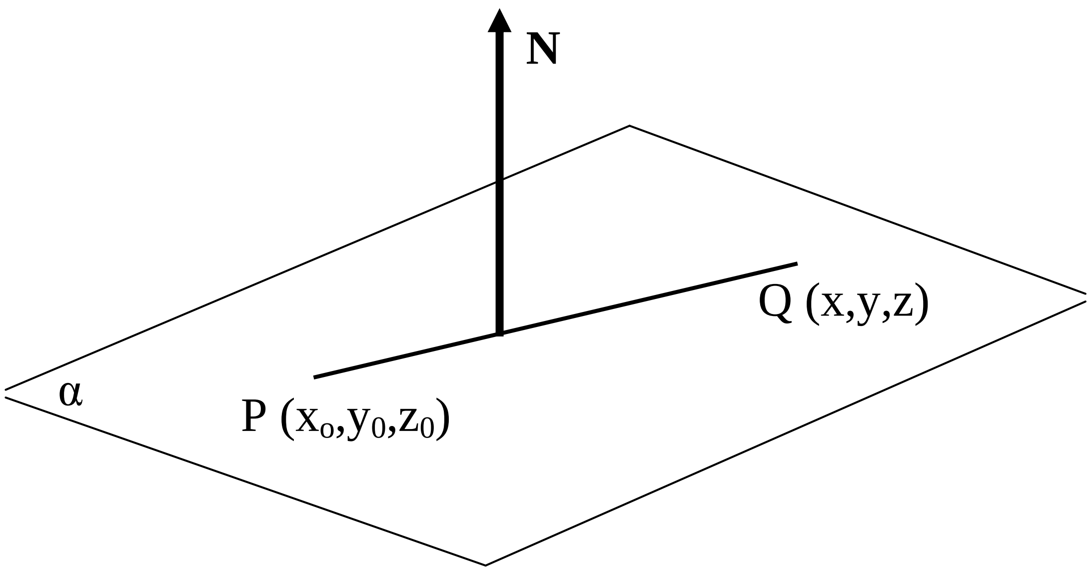
Find the shortest distance from the point $M(1,-2,2)$ to the line

— —

6.5 Planes in Space

6.5.1 Equation of a Plane

Suppose that α is a plane. Point $P(x_0, y_0, z_0)$ and $Q(x, y, z)$ lie on it. If $\overline{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a non-null vector perpendicular (ortoghonal) to α, then ***N*** is perpendicular to PQ.



Thus, $\overline{PQ} \cdot \overline{N} = 0$

$$< x - x_0 , y - y_0 , z - z_0 > \cdot < a, b, c > = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Conclusion:

The equation of a plane can be determined if a point on the plane and a vector orthogonal to the plane are known.

Theorem 6.5 (Equation of a Plane)

The plane through the point $P(x_0, y_0, z_0)$ and with the nonzero normal vector $\mathbf{N} = \langle a, b, c \rangle$ has the equation

Point-normal form:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Standard form:

$ax + by + cz = d$ with $d = ax_0 + by_0 + cz_0$

Example 1*:*

Give an equation for the plane through the point (2, 3, 4) and perpendicular to the vector $\langle -6, 5, -4 \rangle$.

Example 2: (Final 2006/07)

Find the equation of a plane through (2,3,-5) and perpendicular to the line *l:*

Example 3: (Test 1, 2005/06)

Given the plane that contains points *A*(2,1,7), *B*(4,-2,-1), and *C*(3,5,-2). Find:

a) The normal vector to the plane

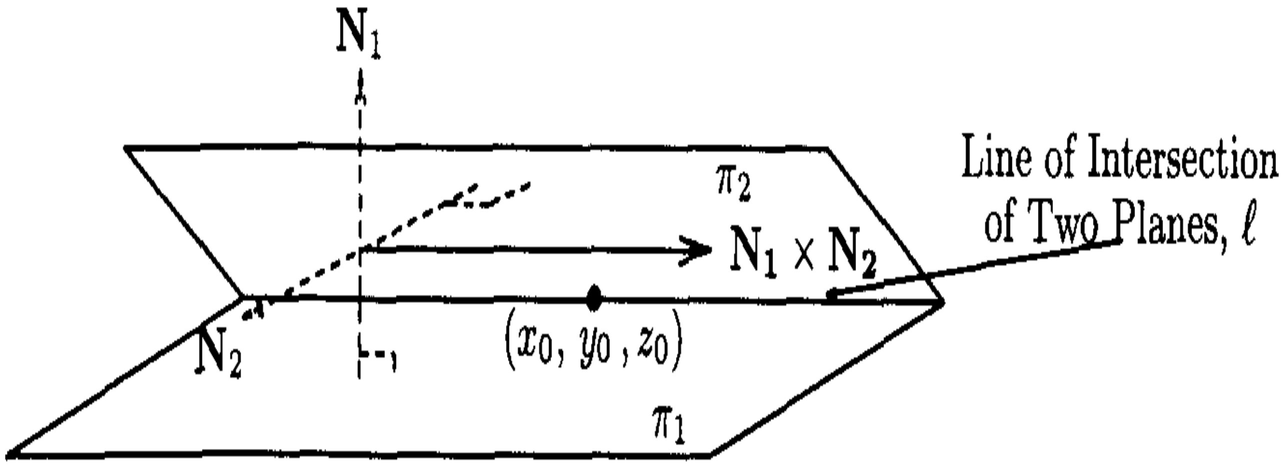
b) The equation of the plane in standard form

Example 4:

Find the parametric equations for the line through the point (5, -3, 2) and perpendicular to the plane $6x+2y-7z=5$.

6.5.2 Intersection Of Two Planes

Intersection of two planes is a line. (L)



To obtain the equation of the intersecting line, we need

1) a point on the line L

2) a vector that is parallel to the line L which is given by

$$= N_1 \times N_2$$

If $\overline{N} = < a, b, c >$, then the equation of the line L is

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \quad \text{(symmetric)}$$

or

$$x = x_0 + at, \; y = y_0 + bt, \; z = z_0 + ct \quad \text{(parametric)}$$

Example 1:

Find the equation of the line passing through $P(2,3,1)$ and parallel to the line of intersection of the planes $x + 2y - 3z = 4$ and $x - 2y + z = 0$.

6.5.3 Angle Between Two Planes

Properties of two planes

(a) An angle between the crossing planes is an angle between their normal vectors.

$$\cos\theta = \frac{N_1 \cdot N_2}{|N_1||N_2|}$$

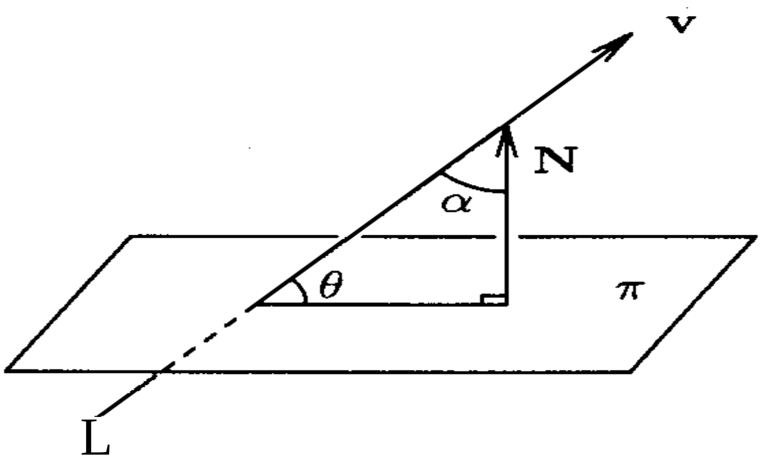
(b) Two planes are parallel if and only if their normal vectors are parallel, $N_1 = \lambda N_2$

(c) Two planes are orthogonal if and only if $N_1 \cdot N_2 = 0$.

Example 1:

Find the angle between plane $3x + 4y = 0$ and plane $2x + y - 2z = 5$.

6.5.4 Angle Between A Line And A Plane



Let α be the angle between the normal vector **N** to a plane π and the line *L*. Then we have

where **v** is vector parallel to *L*.

If θ is the angle between the line *L* and the plane π, then

$$\alpha + \theta = \frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{2} - \alpha$$

and
$$\sin\theta = \sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$$

Therefore, the angle between a line and a plane is

$$\sin\theta = \frac{\mathbf{v}\cdot\mathbf{N}}{|\mathbf{v}||\mathbf{N}|}$$

Example 1: (Final exam, 2006/07)

Calculate the angle between the plane $x - 2y + z = 4$ and the

line —— —— —— .

6.5.5 Shortest Distance Involving Planes

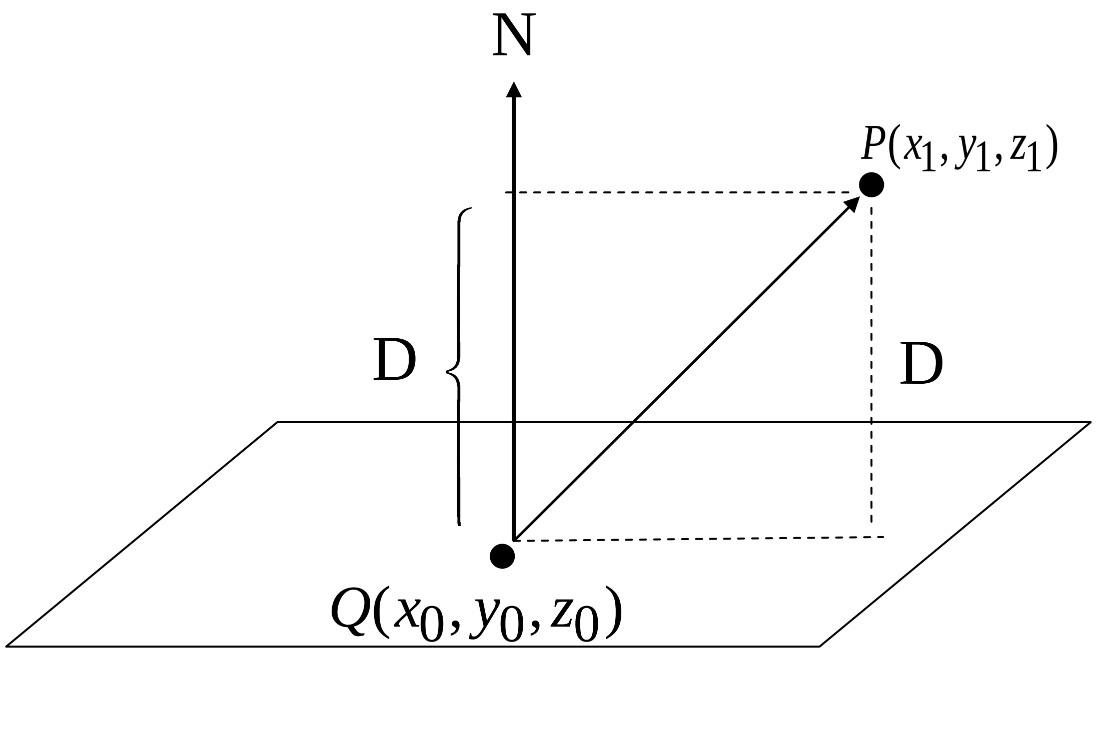
(a) From a Point to a Plane

-Theorem-

The distance D between a point $P(x_1, y_1, z_1)$ and the plane $ax + by + cz = d$ is

$$D = \left| \frac{\mathrm{N} \cdot \overline{QP}}{|\mathrm{N}|} \right| = \left| \frac{ax_1 + by_1 + cz_1 - d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

Where $Q(x_0, y_0, z_0)$ is any point on the plane.



Example 1: (Test 1, 2006/07)

Find the distance D between the point (4, 5, -8) and the plane $2x - 6y + 3z + 4 = 0$.

Example 2:

i. Show that the line

$$\frac{x-1}{3} = \frac{y}{-2} = \frac{z+1}{1}$$

is parallel to the plane $3x - 2y + z = 1$.

ii. Find the distance from the line to the plane in part (a).

<u>(b) Between two parallel planes</u>

The distance between two parallel planes $ax + by + cz = d_1$ and $ax + by + cz = d_2$ is given by

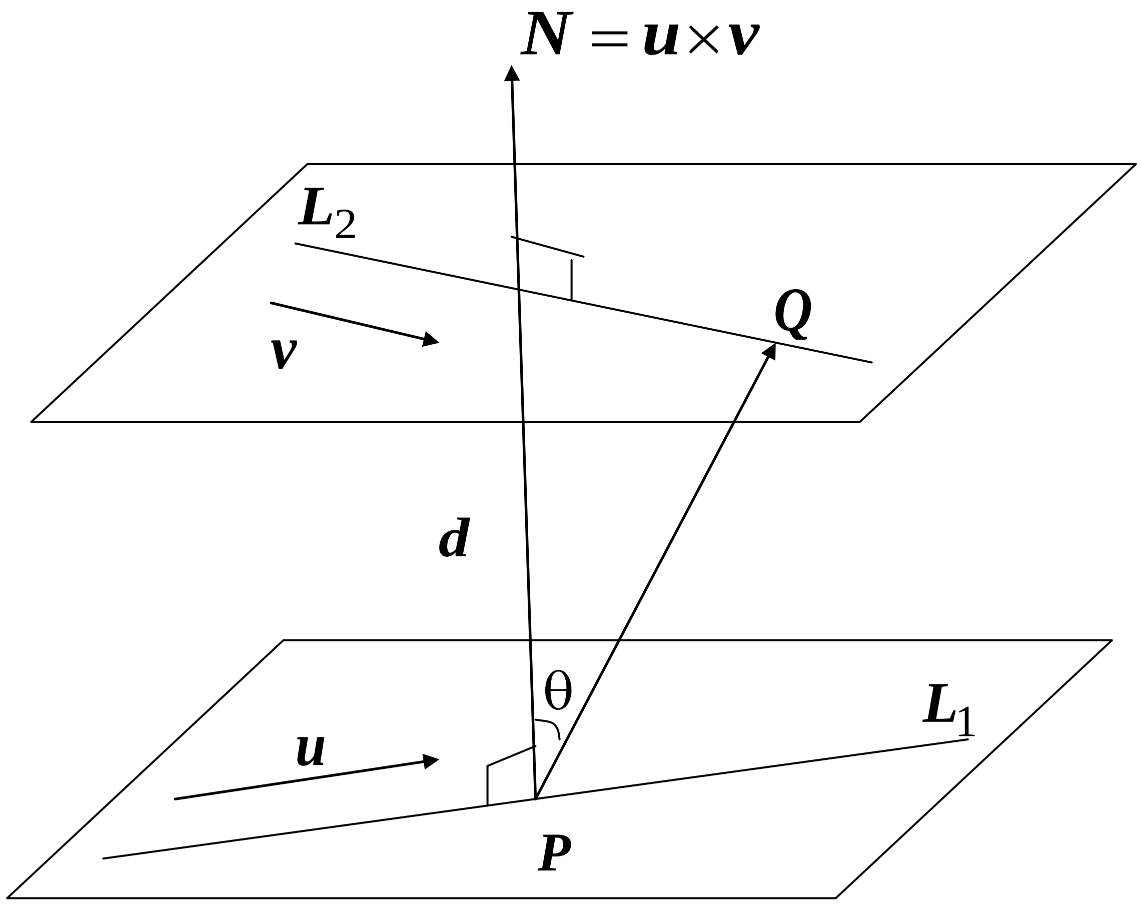
$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

Example 1:

Find the distance between two parallel planes

$x+2y-2z=3$ and $2x+4y-4z=7$.

(c) Between two skewed lines



Assume *L1* and *L2* are skew lines in space containing the points *P* and *Q* and are parallel to vectors **u** and **v** respectively.

Then the shortest distance between *L1* and *L2* is the perpendicular distance between the two lines and its direction is given by a vector normal to both lines.

So, the distance between the two lines is

$$d = \left|\boldsymbol{PQ}\cos\theta\right|$$

$$= \left|\frac{\boldsymbol{N}\cdot\boldsymbol{PQ}}{\left|\boldsymbol{N}\right|}\right| = \left|\frac{(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{PQ}}{\left|\boldsymbol{u}\times\boldsymbol{v}\right|}\right|$$

Example 1:

Find the shortest distance between the skewed lines.

$$l_1 : x = 1+2t,\ y = -1+ t\ ,\ z = 2 + 4t$$

$$l_2 : x = -2+4s,\ y = -3s\ ,\ z = -1+s$$

Example 2:

Find the distance between the lines

$$L_1 : i + 2j + 3k + t(i - k)$$

$$L_2 : x = 0,\ y = 1 + 2t,\ z = 3 + t$$

Example 3:

Find the distance between the lines L_1 through the points A(1, 0, -1) and B(-1, 1, 0) and the line L_2 through the points C(3, 1, -1) and D(4, 5, -2) .