CHAPTER 5 : SERIES

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Review:

Finite and Infinite Sequences

Sequence

A set of numbers written in a particular order

$$u_1, u_2, \ldots, u_n.$$

We sometimes write u_1 for the first term of the sequence, u_2 for the second term and so on. We write the nth term as u_n.

Examples:

1, 3, 5, 9. – finite sequence

1, 2, 3, 4, 5, …, n – finite sequence

1, 1, 2, 3, 5, 8, … - infinite sequence

The Geometric Progression

Example:

$$1, 2, 4, 8, \ldots, 256$$

The finite sequence in equation above is an example of a geometric progression having the general form:

$$a, ax, ax^2, ax^3, \ldots ax^8$$

In this case, $a = 1,\ x = \dfrac{a_r}{a_{r-1}} = \dfrac{a_2}{a_1} = \dfrac{2}{1} = 2$ and

$$u_r = ax^r = 2^r\ ;\ r = 0,1,2,...,8.$$

The Arithmetic Progression

Example:

$$1,\ 3,\ 5,\ 7,\ \ldots,\ 31$$

This finite sequence is an example of an arithmetic progression, because each successive term is given by a sum having the general form:

$$a, a+d, a+2d, a+3d, \ldots, a+(n-1)d.$$

In this case, $a = 1,\ d = a_r - a_{r-1} = 3 - 1 = 2$ and

$$1, 1+2, 1+2\cdot 2, 1+2\cdot 3, \ldots, 1+2\cdot 15.$$

$$u_r = 1 + 2r\ ;\ r = 0,1,2,...,15.$$

5.1 Series

5.1.1 Finite Series

For any sequence of terms $u_1, u_2, \dots, u_n$ we can form a finite series by summing the terms in the sequence up to and including the n^th^ term:

$$S_n = u_1 + u_2 + \dots + u_n = \sum_{r=1}^{n} u_r .$$

$\sum_{r=1}^{n} u_r$ or $\sum u_r$ is the symbol of sum and S_n denotes its n^th^ partial sum.

Example (1):

$$S_n = 1 + 2 + 2^2 + \dots + 2^{n-1}.$$

Evaluate this sum for $n = 1, 2, 3, 4$.

In general, for a geometric series obtained by summing the members of the geometric progression, the sum of the first n[th] is given by:

$$S_n = a + ax + ax^2 + ax^3 + \ldots + ax^{n-1}$$
$$= a\left(\frac{1 - x^n}{1 - x}\right).$$

Example (2):

Find the sum of the geometric series

$$2 + 6 + 18 + 54 + \ldots$$

where there are 6 terms in the series.

5.1.2 Infinite Series

We can also form an infinite series from a sequence by extending the range of the dummy index to an infinite number of terms:

$$S = u_1 + u_2 + u_3 + \ldots = \sum_{r=1}^{\infty} u_r$$

The summation of a finite series will always yield a finite result, but the summation of an infinite series needs careful examination to a finite result, *i.e.* the series converges.

If the sequence S_n is convergent and $\lim_{n\to\infty} S_n = S$ exists as a real number, then the series $\sum a_n$ is called convergent and we write

$$a_1 + a_2 + a_3 + \ldots + a_n = S \text{ or } \sum_{r=1}^{n} a_n = S.$$

The number S is called the sum of the series. If the sequence S_n is divergent then the series is called divergent.

An important example of an infinite series is the geometric series

$$a + ax + ax^2 + ax^3 + \ldots + ax^{n-1} + \ldots = \sum_{n=1}^{\infty} ax^{n-1} \quad a \neq 0.$$

The geometric series is convergent if $|x| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ax^{n-1} = \frac{a}{1-x}.$$

If $|x| \geq 1$, then the geometric series is divergent.

Example (3):

Find the sum of the geometric series

$$5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\ldots$$

Example (4):

Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

Example (5):

Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum?

:: Theorem 1 ::

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0.$

Proof

Let $S_n = a_1 + a_2 + a_3 + \ldots + a_n$. Then $a_n = S_n - S_{n-1}$.

Since $\sum a_n$ is convergent, the sequence S_n is convergent. Let $\lim_{n\to\infty} S_n = S$. Since $n-1 \to \infty$ as $n \to \infty$ we also have $\lim_{n\to\infty} S_{n-1} = S$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \quad S_n - S_{n-1}$$
$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

Note: The converse of the theorem is not true in general. If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\sum_{n=1}^{\infty} a_n$ is convergent.

The Test for Divergence

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

The test for Divergence follows from Theorem 1 because if the series is not divergent, then it is convergent, and so $\lim_{n\to\infty} a_n = 0$.

Example (6):

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges?

Properties of Convergent Series

If $\sum a_n$ and $\sum b_n$ are convergent series and c is a constant then

(1) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$,

(2) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$,

(3) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

Example (7):

Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right)$?

Example (8):

Determine whether the series is convergent or divergent. If the series is convergent find its sum.

(a) $5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\dots$

(b) $\sum_{n=1}^{\infty} 3^{-n}8^{n+1}$

(c) $\sum_{n=1}^{\infty}\frac{1}{n(n+2)}$

(d) $\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n}$

5.2 The Integral and Ratio Tests; The Sum of a Series

In this section, we develop tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum. In some case, however, our methods will enable us to find good estimates of the sum.

The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words

(a) If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: Use this test when $f(x)$ is easy to integrate.

Example (1):

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or diverges.

Example (2):

For what values of P is the series $\sum_{n=1}^{\infty} \frac{1}{n^P}$ convergent?

The Ratio Test

(a) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: The test is very useful in determining whether a given series is absolutely convergent.

Example (3):

Test the convergent of the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$.

Example (4):

Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolutely convergence.

Example (5):

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$..

The Sum of a Series

5.2.1 Sum of Power of 'n' Positive Integers

$$\sum_{r=1}^{n} r = 1+2+3+\ldots+n = \frac{n \ \ n+1}{2}$$

$$\sum_{r=1}^{n} r^2 = 1^2+2^2+3^2+\ldots+n^2 = \frac{1}{6} n \ \ n+1 \ \ \ 2n+1$$

$$\sum_{r=1}^{n} r^3 = 1^3+2^3+3^3+\ldots+n^3 = \left\{ \frac{n \ \ n+1}{2} \right\}^2$$

Example (6):

Evaluate $\sum_{r=1}^{20} r^2$ and $\sum_{r=1}^{25} r^3$.

Example (7):

Evaluate $\sum_{r=1}^{10} \ 2r-1 \ ^2$.

Example (8):

Find the sum for each of the following series:

(a) $1\cdot 3+4\cdot 5+7\cdot 7+\ldots$ to 30 terms

(b) $2^2+4^2+6^2+\ldots+(2n)^2$

5.2.2 Sum of Series of Partial Fraction

The sum of the series $\sum_{r=1}^{n} u_r$ can be determined if the r^th term can be expressed as the difference method. Thus

$$u_r=\frac{1}{c}\left[f(r)-f(r-1)\right], \text{ then}$$

$$\sum_{r=1}^{n} u_r=\frac{1}{k}\left[f(n)-f(0)\right].$$

Note: If we fail to express u_r into this form, $\frac{1}{c}\left[f(r)-f(r-1)\right]$, then this method cannot be used.

Example (9):

Express the following series in terms of r^{th}. Hence by using the difference method, find the sum of the first n terms.

(a) $1\cdot3+3\cdot5+3\cdot7+\ldots$ to 30 terms

(b) $\dfrac{1}{4\cdot5}+\dfrac{1}{5\cdot6}+\dfrac{1}{6\cdot7}+\ldots$

Example (10):

Use the difference method; find the sum of the series $\displaystyle\sum_{r=1}^{n}\frac{2}{r+1\quad r+3}$.

5.3 Power Series

Definition

A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)^n + \ldots$$

in which the center a and the coefficients $a_0, a_1, a_{2,} \ldots, a_n, \ldots$ are constants.

Expansion of Exponent Function

The power series of the exponent function can be written as

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots$$

The expansion is true for all values of *x*. In general,

$$e^{x}=\sum_{n=0}^{\infty}\frac{1}{n!}x^{n}.$$

Example (1):

Given

$$e^{x}=1+x+\frac{1}{2!}x^{2}+\frac{1}{3!}x^{3}+\frac{1}{4!}x^{4}+\ldots+\frac{1}{n!}x^{n}+\ldots$$

Write down the first five terms of the expansion of the following functions

(a) e^{2x}

(b) e^{x-1}

Example (2):

Write down the first five terms on the expansion of the function, $1+x\;^{2}e^{-x}$ in the form of power series.

Expansion of Logarithmic Function

The expansion of logarithmic function can be written as

$$\ln(1+\mathrm{x}) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$$
$$- \frac{1}{6}x^6 + \frac{1}{7}x^7 - \dots$$

The series converges for $-1 < x \leq 1$. Thus the series $\ln\ 1+x$ is valid for $-1 < x \leq 1$.

By assuming x with $-x$, we obtain

$$\ln(1-\mathrm{x}) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5$$
$$- \frac{1}{6}x^6 - \frac{1}{7}x^7 - \dots$$

Thus, this result is true for $-1 < -x \leq 1$ or $-1 \leq x < 1$.

Example (3):

Write down the first five terms of the expansion of the following functions

(a) $\ln\ 1+3x$

(b) $3\ln\ 1-2x^{2}\quad 1+3x$

Example (4):

Find the first four terms of the expansion of the function, $1+x\ ^{2}\ln\ 1+2x\ ^{3}$.

Expansion of Trigonometric Function

The power series for trigonometric functions can be written as

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

Both series are valid for all values of x.

Example (5):

Given

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

Find the expansion of $\cos 2x$ and $\cos 3x$. Hence, by using an appropriate trigonometric identity find the first four terms of the expansion of the following functions:

(a) $\sin^2 x$

(b) $\cos^3 x$

5.4 The Taylor and the Maclaurin Series

Definition 5.9 (TAYLOR AND MACLAURIN SERIES)

If $f(x)$ has a derivatives of all orders at $x = a$, then we call the series as **Taylor's Series** for $f(x)$ about $x = a$ and is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^r}{r!}f^r(a) + \cdots.$$

or

$$f(x+a) = f(a) + x\,f'(a) + \frac{x^2}{2!}f''(a) + \frac{x^3}{3!}f'''(a) + \cdots + \frac{x^r}{r!}f^r(a) + \cdots.$$

In the special case where $a = 0$, this series becomes the **Maclaurin Series** for $f(x)$ and is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x}{r!}f^r(0) + \cdots \quad \diamondsuit$$

Example (1):

Obtain the Taylor series for $f(x) = 3x^2 - 6x + 5$ around the point $x = 1$.

Example (2):

Obtain Maclaurin series expansion for the first four terms of e^x and five terms of $\sin x$. Hence, deduct that Maclaurin series for $e^x \sin x$ is given by

$$x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \dots$$

Example (3):

Use Taylor's theorem to obtain a series expansion of first five terms for $\cos\left(x + \frac{\pi}{3}\right)$. Hence find $\cos 62^\circ$ correct to 4 dcp.

Example (4):

If $y = \ln\cos x$, show that

$$\frac{d^2y}{dx^2} + 1 + \left(\frac{dy}{dx}\right)^2 = 0$$

Hence, by differentiating the above expression several times, obtain the Maclaurin's series of $y = \ln\cos x$ in the ascending power of x up to the term containing x^4.

Finding Limits with Taylor Series and Maclaurin Series.

Example (5):

Find $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.

Example (6):

Evaluate $\lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{3x^4}$.

Evaluating Definite Integrals with Taylor Series and Maclaurin Series.

Example (7):

Use Maclaurin series to approximate the following definite integral.

(a) $\int_0^1 e^{-x^2}\,dx$

(b) $\int_0^1 x\cos(x^3)\,dx$