# CHAPTER 4:

# IMPROPER INTEGRALS

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## 4.1 L'Hopital Rule

If you are doing any limit and you get something in the form 0/0 or ∞/∞, then you should probably try to use L'Hopital rule. The basic idea of L'Hospital rule is simple.

Consider the limit

$$\lim_{x\to a}\frac{f(x)}{g(x)}.$$

If both the numerator and the denominator are finite at *a* and $g(a)\neq 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f(a)}{g(a)}.$$

*Example 1:*

$$\lim_{x\to 3}\frac{x^2+1}{x+2}=\frac{10}{5}=2.$$

But what happens if both the numerator and the denominator tend to zero???

It is not clear what the limit is. In fact, depending on what functions $f(x)$ and $g(x)$ are, the limit can be anything at all!!!

### *4.1.1 L'Hopital Rule for 0/0*

Suppose $\lim f(x) = \lim g(x) = 0$. Then

1. If $\lim \dfrac{f'(x)}{g'(x)} = L$,

$$\text{then } \lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} = L.$$

2. If $\lim \dfrac{f'(x)}{g'(x)}$ tends to $+\infty$ or $-\infty$ in the limit, then so does $\lim \dfrac{f(x)}{g(x)}$.

*Example 2:*

Find $\lim_{x\to 0}\frac{\sin x}{x}$ by L'Hopital rule.

*Example 3:*

Find $\lim_{x\to 1}\frac{2\ln x}{x-1}$.

*Example 4:*

Find $\lim_{x\to 1}\frac{e^x-1}{x^2}$.

*Example 5:*

Find $\lim_{x\to 0}\frac{1-\cos x}{x}$.

Note: If the numerator and the denominator both tend to $+\infty$ or $-\infty$, L'Hopital rule still applies.

### ***4.1.2 L'Hopital Rule for*** $\infty/\infty$

Suppose $\lim f(x)$ and $\lim g(x)$ are both infinite. Then

1. If $\lim \frac{f'(x)}{g'(x)} = L$,

$$\text{then } \lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} = L.$$

2. If $\lim \dfrac{f'(x)}{g'(x)}$ tends to $+\infty$ or $-\infty$ in the limit, then so does $\lim \dfrac{f(x)}{g(x)}$.

*Example 6:*

Find $\lim\limits_{x\to\infty} \dfrac{x}{e^x}$.

*Example 7:*

Find $\lim\limits_{x\to\infty} \dfrac{\ln\left(\ln x^{1000}\right)}{\ln x}$.

## 4.2 Improper Integrals

The definite integral

$$\int_a^b f(x)\,dx$$

is known as *improper integral* if either

1) one or both limits are infinite, or
2) $f(x)$ is undefined at certain points on/in the interval.

Note: We called case: 1) as Type I
2) as Type II

### *4.2.1 Improper Integral Type 1*

1) If $f(x)$ is continuous in the interval $[a,\infty)$,

$$\text{then} \int_a^{\infty} f(x)\,dx = \lim_{T\to\infty} \int_a^{T} f(x)\,dx.$$

2) If *f(x)* is continuous in the interval (-∞, *b*],

$$\text{then} \int_{-\infty}^{b} f(x)\, dx = \lim_{T \to -\infty} \int_{T}^{b} f(x)\, dx.$$

Note: the improper integrals in 1) and 2) is said to *converge* if the limit exists and *diverge* if the limit does not exist.

3) If *f(x)* is continuous in the interval (-∞,∞),

$$\text{then} \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{c} f(x)\, dx + \int_{c}^{\infty} f(x)\, dx$$

with any real number *c*.

Note: the improper integrals in 3) is said to *converge* if both terms converge and *diverge* if either term diverges.

*Example 8:*

Determine whether the following integral are convergent or divergent:

1) $\int_1^\infty \frac{1}{x}\,dx$

2) $\int_0^\infty xe^{-x}\,dx$

3) $\int_{-\infty}^\infty \frac{x}{1+x^2}\,dx$

*Example 9:*

For what values of $p$ is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

### *4.2.2 Improper Integral Type 2*

1) If $f\left(x\right)$ is continuous on [$a$,$b$), and discontinuous at $b$, then

$$\int_a^b f(x)\,dx = \lim_{T\to b^-} \int_a^T f(x)\,dx.$$

2) If $f(x)$ is continuous on ($a$,$b$], and discontinuous at $a$, then

$$\int_a^b f(x)\,dx = \lim_{T\to a^+} \int_T^b f(x)\,dx.$$

Note: the improper integrals in 1) and 2) is said to *converge* if the limit exists and *diverge* if the limit does not exist.

3) If $f(x)$ has discontinuity at $c$, where $a < c < b$ , then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Note: the improper integrals in 3) is said to *converge* if both terms converge and *diverge* if either term diverges.

*Example 10:*

Determine whether $\int_1^9 \frac{1}{\sqrt[3]{x-9}}\,dx$ converge or diverge.

*Example 11:*

Find $\int_0^3 \frac{1}{x-1}\,dx$ if possible.