

CHAPTER 2

DIFFERENTIATION

2.1 Differentiation of hyperbolic functions

2.2 Differentiation of inverse trigonometric functions

2.3 Differentiation of inverse hyperbolic functions

***Recall: Methods of differentiation**

- Chain rule
- Product differentiation
- Quotient differentiation
- Implicit differentiation

2.1 Differentiation of Hyperbolic Functions

Recall: Definition:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

Derivatives of hyperbolic functions

Example 2.1: Find the derivatives of

(a) $\sinh x$

(b) $\cosh x$

(c) $\tanh x$

Solution:

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx} \sinh x &= \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) \\ &= \frac{1}{2} (e^x + e^{-x}) = \cosh x \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{d}{dx} \cosh x &= \frac{d}{dx} \left(\frac{e^x + e^{-x}}{2} \right) \\ &= \frac{1}{2} (e^x - e^{-x}) = \sinh x \end{aligned}$$

$$(c) \frac{d}{dx} \tanh x = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)$$

Using quotient diff:

$$= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{(e^x + e^{-x})^2}$$

$$= \frac{4}{(e^x + e^{-x})^2} = \left(\frac{2}{(e^x + e^{-x})} \right)^2$$

$$= \left(\frac{1}{\cosh x} \right)^2 = \operatorname{sech}^2 x$$

Using the same methods, we can obtain the derivatives of the other hyperbolic functions and these gives us the standard derivatives.

Standard Derivatives

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{cosech} x$	$-\operatorname{cosech} x \coth x$
$\coth x$	$-\operatorname{cosech}^2 x$

Example 2.2:

1. Find the derivatives of the following functions:

a) $y = \frac{x^3}{\sinh x}$

b) $y = \tanh^3 2x$

c) $y = e^{\cosh 4x^2}$

2. Find the derivatives of the following functions:

(a) $y = \cosh(3x)$

(b) $r = \sinh(2t^2 - 1)$

(c) $g(x) = (x - 1)^3 \operatorname{sech}^2 x$

(d) $y = \tanh(\ln x)$

3. (Implicit differentiation)

Find $\frac{dy}{dx}$ from the following expressions:

(a) $x = y^2 \sinh 4x + \cosh y$

(b) $y = \tanh(x + y)$

2.2 Differentiation Involving Inverse Trigonometric Functions

Recall: Definition of inverse trigonometric functions

Function	Domain	Range
$\sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$\cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$\tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$\sec^{-1} x$	$ x \geq 1$	$0 \leq y < \frac{\pi}{2} \cup \frac{\pi}{2} < y < \pi$
$\cot^{-1} x$	$-\infty \leq x \leq \infty$	$0 < y < \pi$
$\operatorname{cosec}^{-1} x$	$ x \geq 1$	$-\frac{\pi}{2} < y < 0 \cup 0 < y < \frac{\pi}{2}$

Derivatives of Inverse Trigonometric Functions

Standard Derivatives:

$$1. \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$2. \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$3. \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$4. \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$5. \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$6. \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

2.2.1 Derivatives of $y = \sin^{-1} x$. (proof)

Recall: $y = \sin^{-1} x \Leftrightarrow x = \sin y$

for $x \in [-1, 1]$ and $y \in [-\pi/2, \pi/2]$.

Because the sine function is differentiable on $[-\pi/2, \pi/2]$, the inverse function is also differentiable.

To find its derivative we proceed implicitly:

Given $\sin y = x$. Differentiating w.r.t. x :

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

$$\cos y \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$$

Since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $\cos y \geq 0$, so

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

Example 2.3:

1. Differentiate each of the following functions.

(a) $f(x) = \tan^{-1} \sqrt{x}$

(b) $g(t) = \sin^{-1}(1-t)$

(c) $h(x) = \sec^{-1} e^{2x}$

2. Find the derivative of:

(a) $y = (\tan^{-1} x^2)^4$

(b) $f(x) = \ln(\sin^{-1} 4x)$

3. Find the derivative of $y = \tan^{-1}(\tan(3t^2 - 1))$.

4. Find the derivative $\frac{dy}{dx}$ if

(a) $x \tan^{-1} y = x^2 + y$

(b) $\sin^{-1}(xy) + \frac{\pi}{2} = \cos^{-1} y$

Summary

If u is a differentiable function of x , then

$$1. \frac{d}{dx}(\sin^{-1} u) = \frac{1}{1-u^2} \frac{du}{dx}$$

$$2. \frac{d}{dx}(\cos^{-1} u) = \frac{-1}{1-u^2} \frac{du}{dx}$$

$$3. \frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2} \frac{du}{dx}$$

$$4. \frac{d}{dx}(\cot^{-1} u) = \frac{-1}{1+u^2} \frac{du}{dx}$$

$$5. \frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx}$$

$$6. \frac{d}{dx}(\csc^{-1} u) = \frac{-1}{|u|\sqrt{u^2-1}} \frac{du}{dx}$$

2.3 Derivatives of Inverse hyperbolic Functions

Recall: Inverse Hyperbolic Functions

Function	Domain	Range
$y = \sinh^{-1} x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$y = \cosh^{-1} x$	$[1, \infty)$	$[0, \infty)$
$y = \tanh^{-1} x$	$(-1, 1)$	$(-\infty, \infty)$
$y = \coth^{-1} x$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \operatorname{sech}^{-1} x$	$(0, 1]$	$[0, \infty)$
$y = \operatorname{cosech}^{-1} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Function	Logarithmic form
$y = \sinh^{-1} x$	$\ln(x + \sqrt{x^2 + 1})$
$y = \cosh^{-1} x$	$\ln(x + \sqrt{x^2 - 1})$
$y = \tanh^{-1} x$	$\frac{1}{2} \ln\left(\frac{1+x}{1-x}\right); x < 1$

2.3.1 Proof: ($\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$)

Recall: $y = \sinh^{-1} x \Leftrightarrow x = \sinh y$

To find its derivative we proceed implicitly:

➤ Given $x = \sinh y$. Differentiating w.r.t. x :

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sinh y)$$

$$1 = \cosh y \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cosh y}$$

➤ Since $-\infty < y < \infty$, $\cosh y \geq 0$, so using the identity

$$\cosh^2 y - \sinh^2 y = 1:$$

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}$$

$$\therefore \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1 + x^2}}$$

➤ Other ways to obtain the derivatives are:

(a) $y = \sinh^{-1} x \Leftrightarrow x = \sinh y$ then

$$x = \frac{e^y - e^{-y}}{2}. \text{ Hence, find } \frac{dy}{dx}.$$

$$(b) \ y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right).$$

$$\text{Hence, find } \frac{dy}{dx}.$$

Standard Derivatives

Function, y	Derivatives, $\frac{dy}{dx}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2 + 1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2 - 1}}; x > 1$
$\tanh^{-1} x$	$\frac{1}{1 - x^2}; x < 1$
$\coth^{-1} x$	$\frac{1}{1 - x^2}; x > 1$
$\operatorname{sech}^{-1} x$	$-\frac{1}{x\sqrt{1 - x^2}}; 0 < x < 1$
$\operatorname{cosech}^{-1} x$	$\frac{1}{ x \sqrt{1 + x^2}}; x \neq 0$

Generalised Form

$y = f(u);$ $u = g(x)$	$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$\sinh^{-1} u$	$\frac{1}{\sqrt{u^2 + 1}} \frac{du}{dx}$
$\cosh^{-1} u$	$\frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}; u > 1$
$\tanh^{-1} u$	$\frac{1}{1 - u^2} \frac{du}{dx}; u < 1$
$\coth^{-1} u$	$\frac{1}{1 - u^2} \frac{du}{dx}; u > 1$
$\operatorname{sech}^{-1} u$	$-\frac{1}{u\sqrt{1 - u^2}} \frac{du}{dx}; 0 < u < 1$
$\operatorname{cosech}^{-1} u$	$\frac{1}{ u \sqrt{1 + u^2}} \frac{du}{dx}; u \neq 0$

Example 2.4: Find the derivatives of

(a) $y = \sinh^{-1}(1 - 3x)$

(b) $y = \cosh^{-1}\left(\frac{1}{x}\right)$

(c) $y = e^x \operatorname{sech}^{-1} x$

(d) $y = \sinh^{-1}(\tan 3x)$

(e) $f(t) = \frac{\tanh^{-1} t^2}{1 - \sec t}$

(f) $y = \sqrt{\coth^{-1} u}$

(g) $y = \cos 4x \cosh^{-1} 4x$

(h) $y^3 - \sinh^{-1} xy = 0$