CHAPTER 1

FURTHER TRANSCENDENTAL FUNCTIONS

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1.2.2 Inverse Trigonometric Identities

1.2.3 Inverse Hyperbolic Functions

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1.1.1 Definition of Hyperbolic Functions

❖**Hyperbolic Sine,** pronounced **"shine".**

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

❖**Hyperbolic Cosine,** pronounced **"cosh".**

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

❖**Hyperbolic Tangent,** pronounced **"tanh".**

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \equiv \frac{e^{2x} - 1}{e^{2x} + 1}$$

❖**Hyperbolic Secant,** pronounced **"shek".**

$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

❖**Hyperbolic Cosecant,** pronounced **"coshek".**

$$\text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

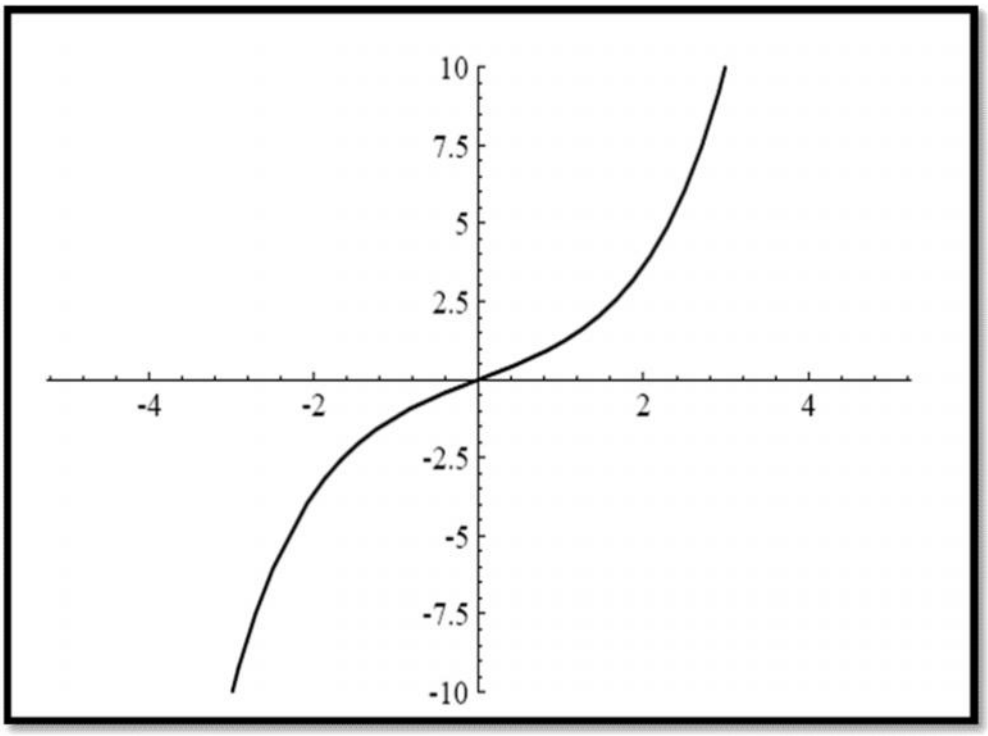
❖**Hyperbolic Cotangent,** pronounced **"coth".**

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

1.2 Graphs of Hyperbolic Functions

Since the hyperbolic functions depend on the values of e^x and e^{-x}, its graphs is a combination of the exponential graphs.

(i) Graph of sinh *x*



From the graph, we see

(i) $\sinh 0 = 0$.

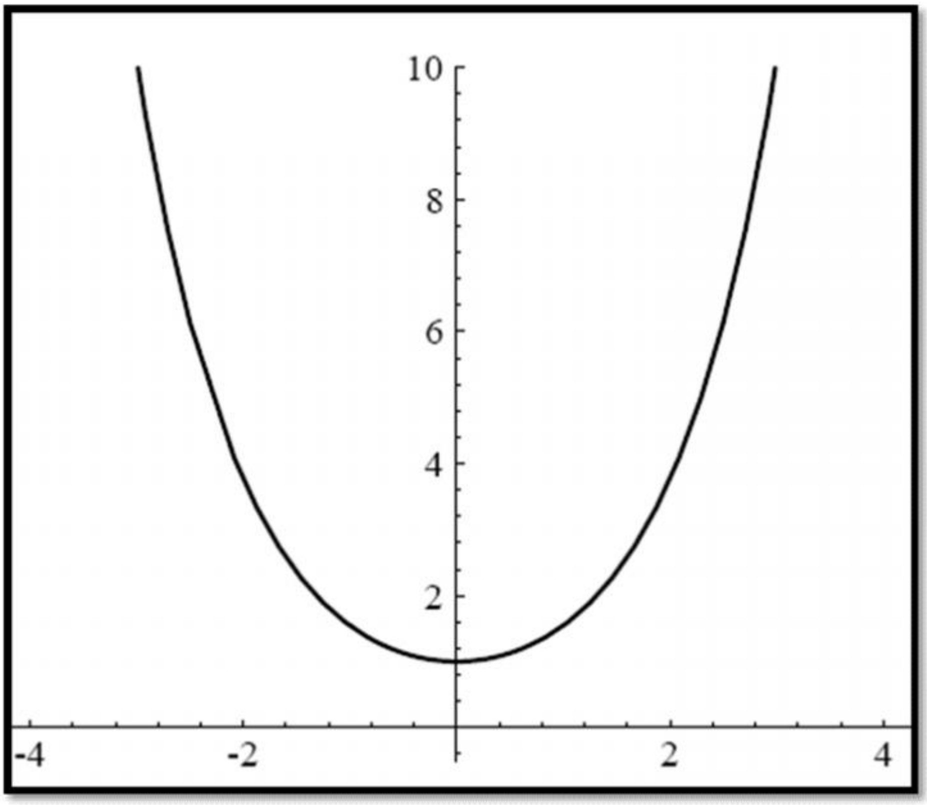
(ii) The domain is all real numbers

(iii) The curve is symmetrical about the origin, i.e.

$\sinh(-x) = -\sinh x$

(iv) It is an increasing one-to-one function.

(ii) Graph of cosh x



We see from the graph of $y = \cosh x$ that:

(i) $\cosh 0 = 1$

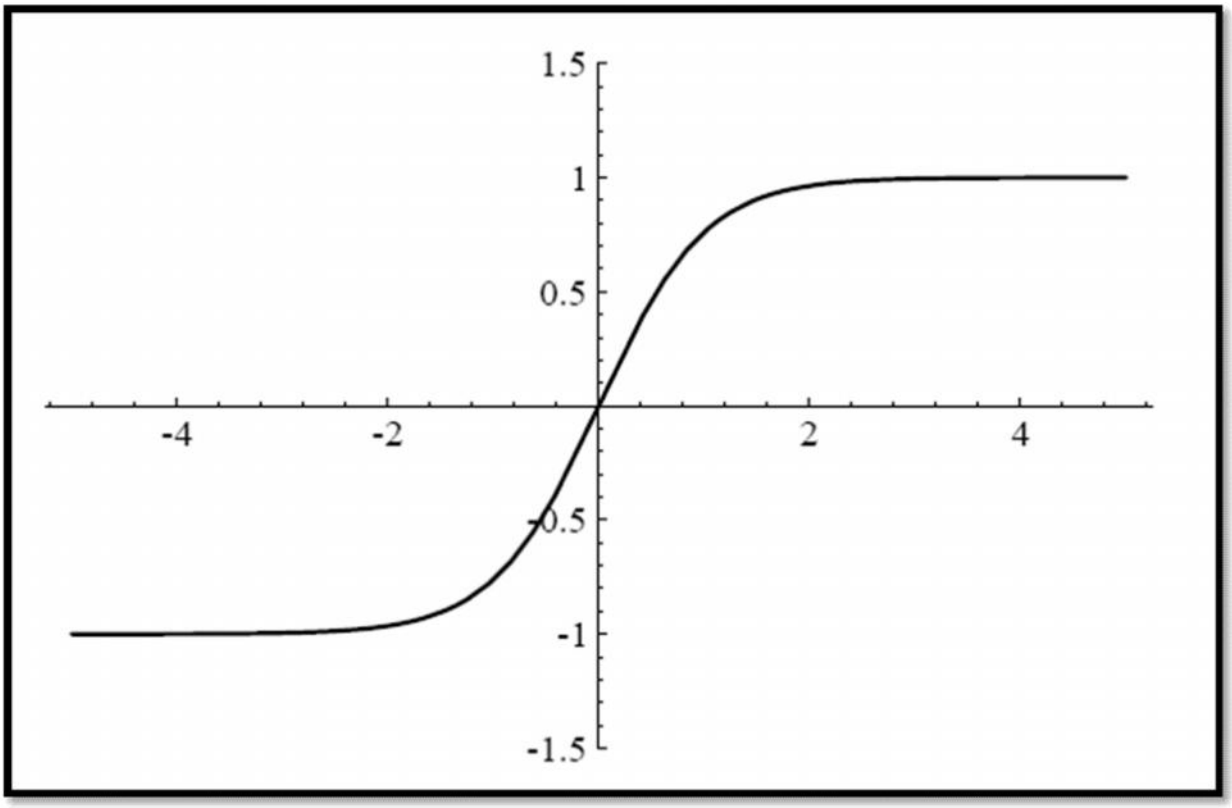
(ii) The domain is all real numbers.

(iii) The value of $\cosh x$ is never less than 1.

(iv) The curve is symmetrical about the y-axis, i.e.
$\cosh(-x) = \cosh x$

(v) For any given value of $\cosh x$, there are two values of x.

(ii) Graph of tanh x



We see

(i) $\tanh 0 = 0$

(ii) $\tanh x$ always lies between $y = -1$ and $y = 1$.

(iii) $\tanh(-x) = -\tanh x$

(iv) It has horizontal asymptotes $y = \pm 1$.

1.1.3 Hyperbolic Identities

For every identity obeyed by trigonometric functions, there is a corresponding identity obeyed by hyperbolic functions.

1. $\cosh^2 x - \sinh^2 x = 1$
2. $1 - \tanh^2 x = \operatorname{sec}h^2 x$
3. $\coth^2 x - 1 = \operatorname{cos}ech^2 x$
4. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
5. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
6. $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
7. $\sinh 2x = 2 \sinh x \cosh x$
8. $\cosh 2x = \cosh^2 x + \sinh^2 x$
   $= 2\cosh^2 x - 1$
   $= 2\sinh^2 x + 1$
9. $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

Some of the hyperbolic identities follow exactly the trig. identities; others have a difference in sign.

Trig. Identities	Hyperbolic Identities
$\sec\theta \equiv \dfrac{1}{\cos\theta}$ $\operatorname{cosec}\theta \equiv \dfrac{1}{\sin\theta}$ $\cot\theta \equiv \dfrac{1}{\tan\theta}$	$\operatorname{sech}\theta = \dfrac{1}{\cosh\theta}$ $\operatorname{cosech}\theta = \dfrac{1}{\sinh\theta}$ $\coth\theta = \dfrac{1}{\tanh\theta}$
$\cos^2\theta + \sin^2\theta \equiv 1$ $1 + \tan^2\theta \equiv \sec^2\theta$ $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$	$\cosh^2\theta - \sinh^2\theta \equiv 1$ $1 - \tanh^2\theta \equiv \operatorname{sech}^2\theta$ $\coth^2\theta - 1 \equiv \operatorname{cosech}^2\theta$
$\sin 2A \equiv 2\sin A\cos A$ $\cos 2A \equiv \cos^2 A - \sin^2 A$ $\equiv 1 - 2\sin^2 A$ $\equiv 2\cos^2 A - 1$	$\sinh 2A \equiv 2\sinh A\cosh A$ $\cosh 2A \equiv \cosh^2 A + \sinh^2 A$ $\equiv 1 + 2\sinh^2 A$ $\equiv 2\cosh^2 A - 1$

Examples 1.1

1. By <u>using definition</u> of hyperbolic functions,

   a) Evaluate sinh(-4) to four decimal places.

   b) Show that $2\cosh^2 x - 1 = \cosh 2x$

2. a) By using identities of hyperbolic functions, show that

   $$\frac{1-\tanh^2 x}{1+\tanh^2 x} = \operatorname{sech} 2x$$

   b) Solve the following for x, giving your answer in 4dcp.

   $\cosh 2x = \sinh x + 1$

3. Solve for x if given $2\cosh x - \sinh x = 2$.

4. a) By using definition of hyperbolic functions, proof that $\cosh^2 x - \sinh^2 x = 1$

   b) Solve $\cosh x = 4 - \sinh x$. Use 4 dcp.

1.2 Inverse Functions

Definition 1.2 (Inverse Functions)

If $f: X \rightarrow Y$ is a one-to-one function with the domain X and the range Y, then there exists an inverse function,

$$f^{-1}: Y \rightarrow X$$

where the domain is Y and the range is X such that

$$y = f(x) \Leftrightarrow x = f^{-1}(y)$$

Thus, $f^{-1}(f(x)) = x$ for all values of x in the domain f.

Note:

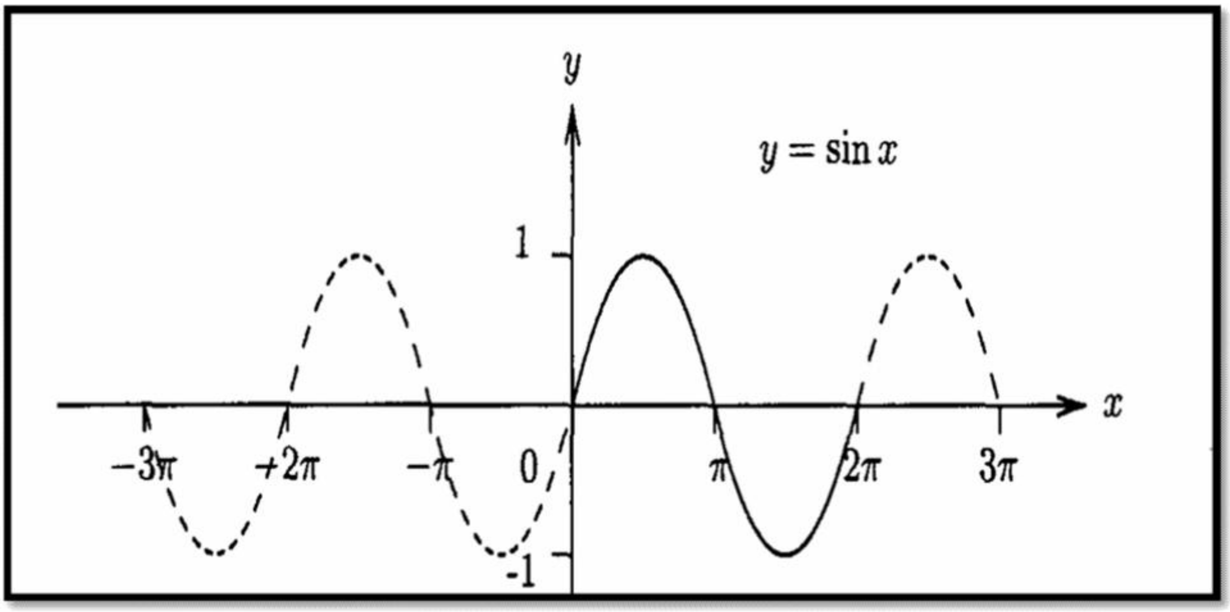
The graph of inverse function is reflections about the line $y = x$.

1.2.1 Inverse Trigonometric Functions

Trigonometric functions are **periodic** hence they are **not one-to one**. However, if we **restrict the domain** to a chosen interval, then the restricted function is one-to-one and invertible.

(i) Inverse Sine Function

Look at the graph of $y = \sin x$ shown below



The function $f(x) = \sin x$ is not one to one. But if the domain is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then *f(x)* is one to one.

Definition:

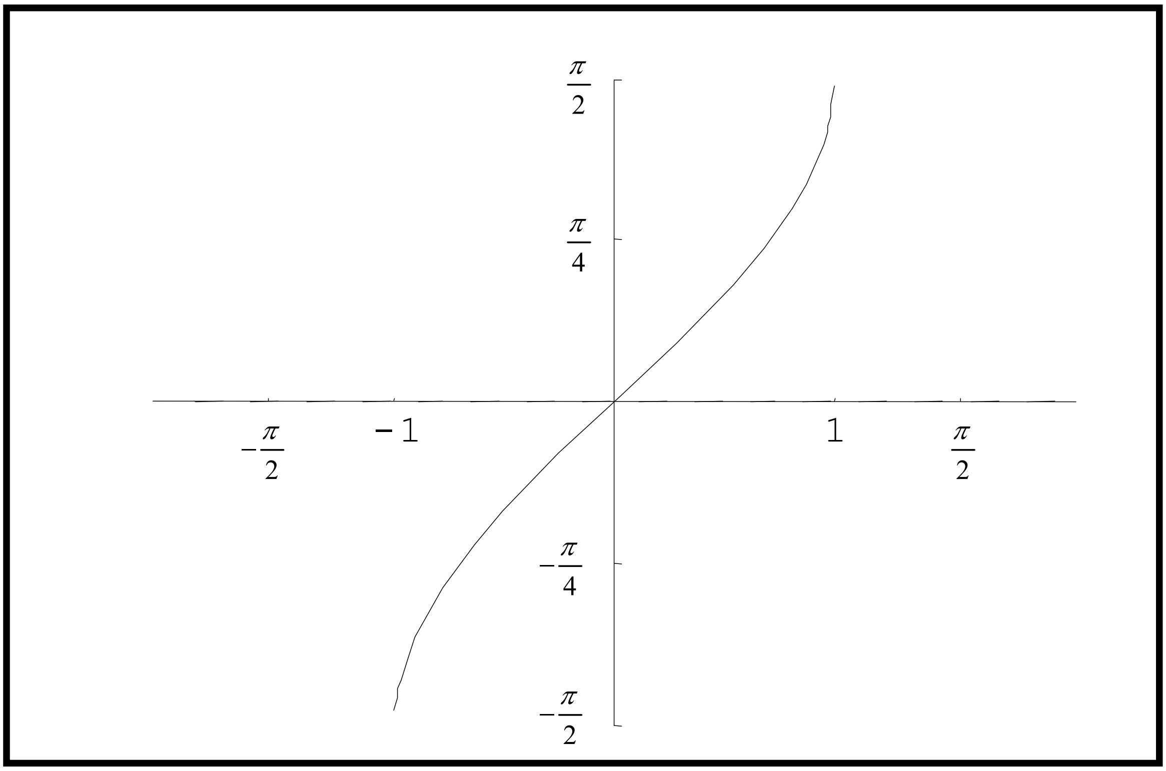
The inverse sine function is defined as

$$y = \sin^{-1} x \Leftrightarrow x = \sin y$$

where $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$ and $-1 \le x \le 1$.

The function $\sin^{-1} x$ is sometimes written as arcsin x.

The graph of $y = \sin^{-1} x$ is shown below

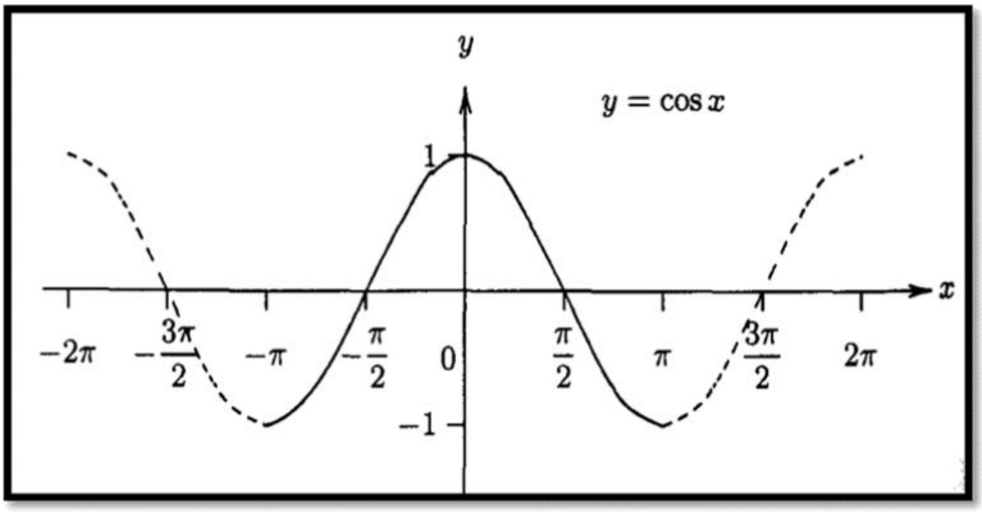


$f(x) = \sin^{-1} x$

$f(x) = \arcsin x$

ii) Inverse Cosine Function

Look at the graph of $y = \cos x$ shown below



The function $f(x) = \cos x$ is not one to one. But if the domain is restricted to $[0, \pi]$, then *f(x)* is one to one.

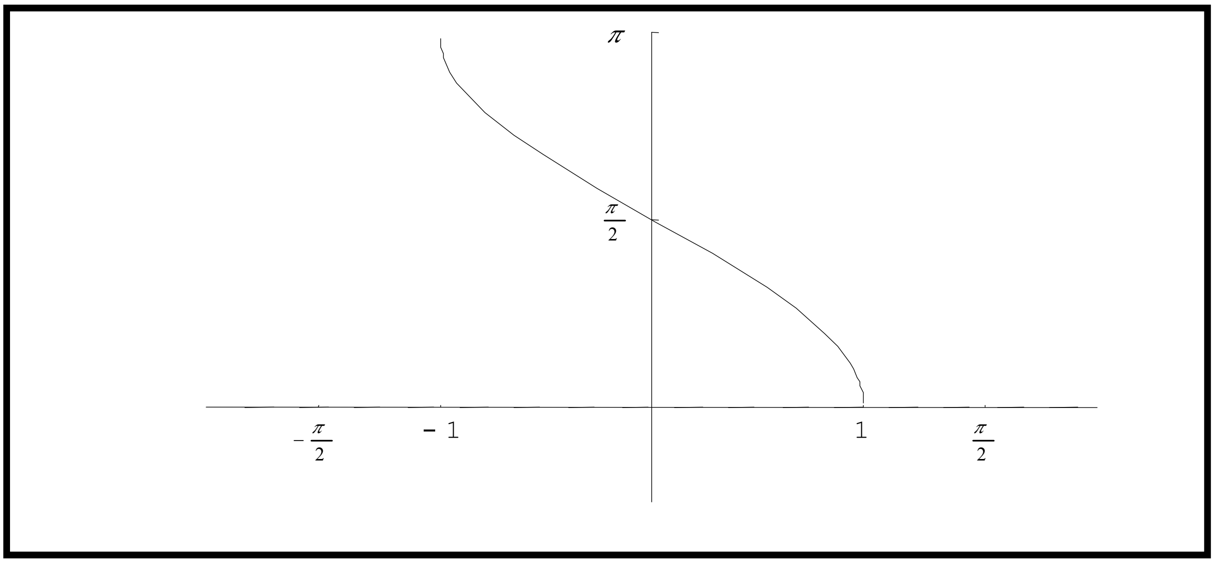
Definition:

The inverse cosine function is defined as

$$y = \cos^{-1} x \Leftrightarrow x = \cos y$$

where $0 \leq y \leq \pi$ and $-1 \leq x \leq 1$.

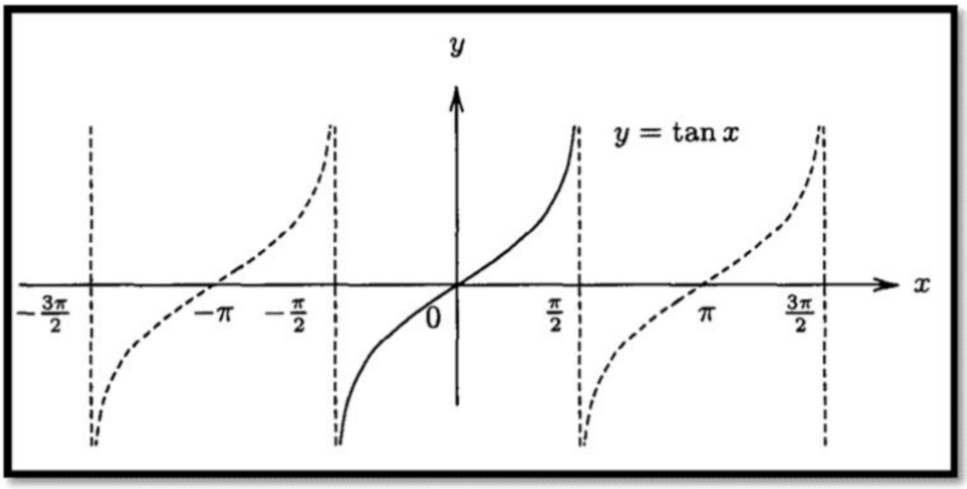
The graph of $y = \cos^{-1} x$ is shown below



$$f(x) = \cos^{-1} x$$
$$f(x) = \arccos x$$

(iii) Inverse Tangent Function

Look at the graph of $y = \tan x$ shown below



The function $f(x) = \tan x$ is not one to one. But if the domain is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then *f(x)* is one to one.

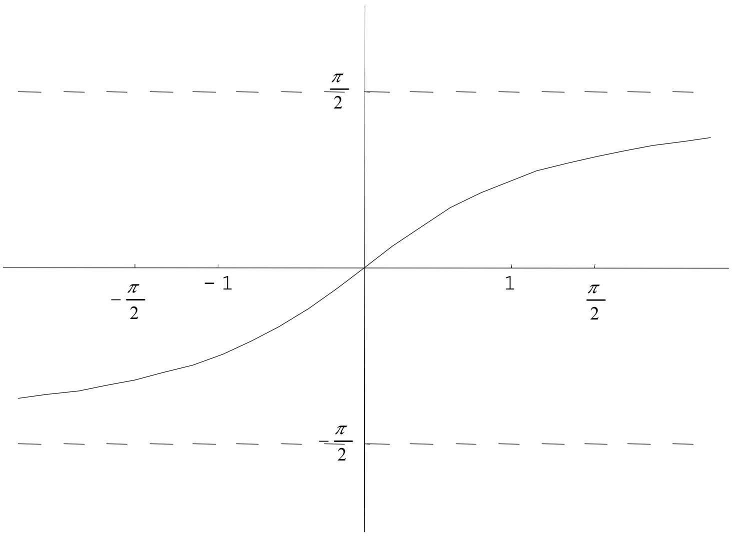
Definition:

The inverse tangent function is defined as

$$y = \tan^{-1} x \iff x = \tan y$$

where $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$ and $-\infty \le x \le \infty$.

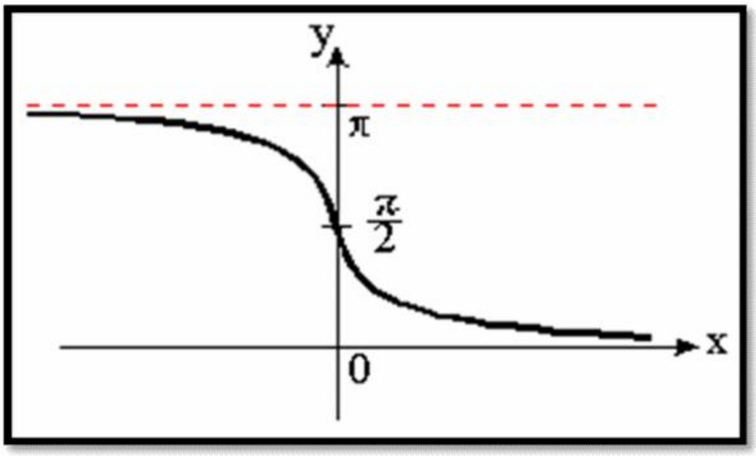
The graph of $y = \tan^{-1} x$ is shown below



$$f(x) = \tan^{-1} x$$

$$f(x) = \arctan x$$

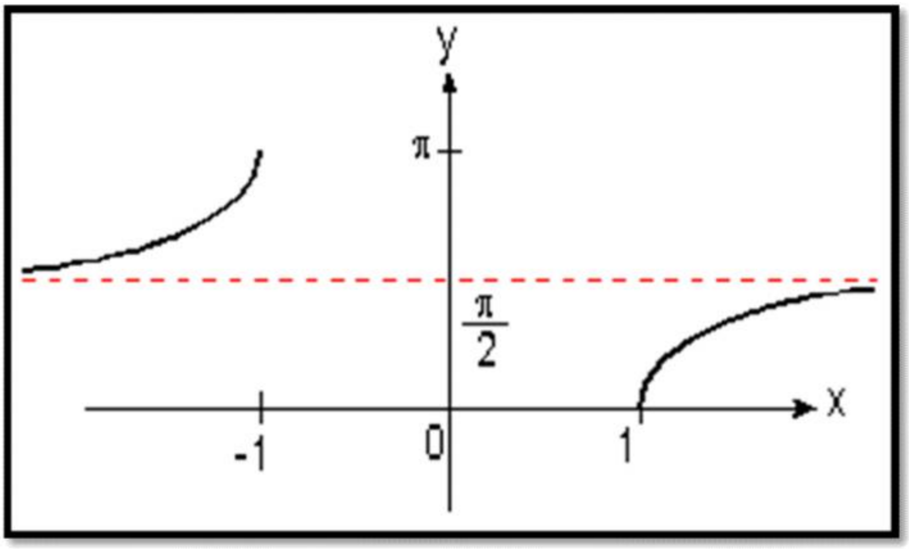
(iv) Inverse Cotangent Function
Domain:
Range:



(v) Inverse Secant Function

Domain:

Range:



(vi) Inverse Cosecant Function

Domain:

Range:

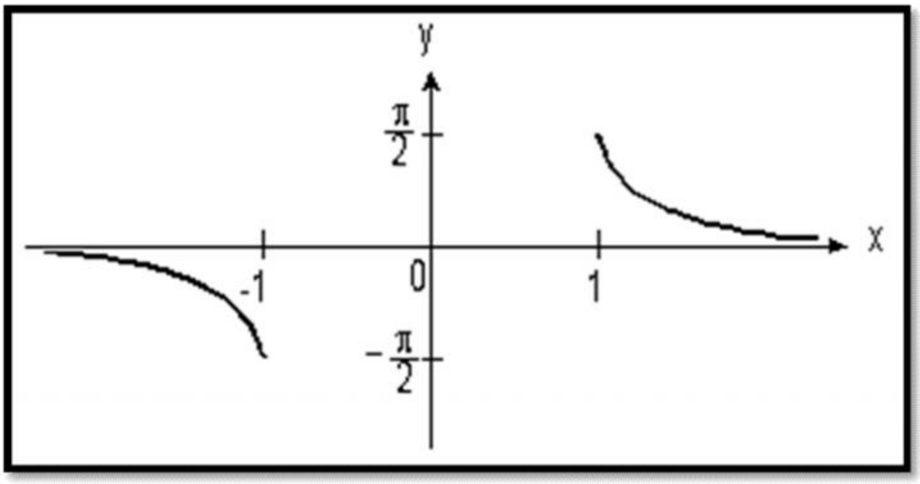


Table of Inverse Trigonometric Functions

Functions	Domain	Range	Quadrants
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	I & IV
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	I & II
$y = \tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	I & IV
$y = \csc^{-1} x$	$\lvert x \rvert \geq 1$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$	I & IV
$y = \sec^{-1} x$	$\lvert x \rvert \geq 1$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$	I & II
$y = \cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$	I & II

- It is easier to remember the restrictions on the domain and range if you do so in terms of quadrants.
- $\sin^{-1} x \neq \frac{1}{\sin x}$ whereas $(\sin x)^{-1} = \frac{1}{\sin x}$.

1.2.2 Inverse Trigonometric Identities

The definition of the inverse functions yields several formulas.

Inversion formulas

$\sin(\sin^{-1} x) = x$	for $-1 \le x \le 1$
$\sin^{-1}(\sin y) = y$	for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$\tan(\tan^{-1} x) = x$	for all x
$\tan^{-1}(\tan y) = y$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

- These formulas are valid only on the specified domain

Basic Relation

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$	for $0 \le x \le 1$
$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$	for $0 \le x \le 1$
$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$	for $0 \le x \le 1$

Negative Argument Formulas

$\sin^{-1}(-x) = -\sin^{-1} x$	$\sec^{-1}(-x) = \pi - \sec^{-1} x$
$\tan^{-1}(-x) = -\tan^{-1} x$	$\csc^{-1}(-x) = -\csc^{-1} x$
$\cos^{-1}(-x) = \pi - \cos^{-1} x$	$\cot^{-1}(-x) = \pi - \cot^{-1} x$

Reciprocal Identities

$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$	for $\lvert x \rvert \geq 1$
$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$	for $\lvert x \rvert \geq 1$
$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$	for all x

Examples 1.2:

1.Evaluate the given functions.

(i) $\sin(\sin^{-1} 0.5)$ (ii) $\sin(\sin^{-1} 2)$

(iii) $\sin^{-1}(\sin 0.5)$ (iv) $\sin^{-1}(\sin 2)$

2. Evaluate the given functions.

(i) $\text{arcsec}(-2)$ (ii) $\csc^{-1}(\sqrt{2})$

(iii) $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

3. For $-1 \le x \le 1$, show that

(i) $\sin^{-1}(-x) = -\sin^{-1} x$

(ii) $\cos(\sin^{-1} x) = \sqrt{1-x^2}$

1.2.3 Inverse Hyperbolic Functions

The three basic inverse hyperbolic functions are $\sinh^{-1} x$, $\cosh^{-1} x$, and $\tanh^{-1} x$.

Definition (*Inverse Hyperbolic Function*)

$$y = \sinh^{-1} x \quad \Leftrightarrow \quad x = \sinh y \quad \text{for all } x \text{ and } y \in \Re$$

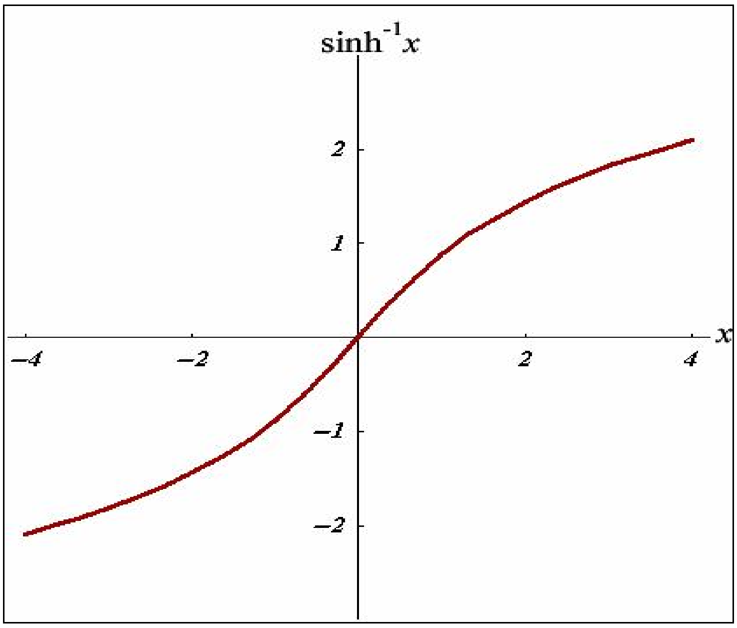
$$y = \cosh^{-1} x \quad \Leftrightarrow \quad x = \cosh y \quad \text{for } x \geq 1 \text{ and } y \geq 0$$

$$y = \tanh^{-1} x \quad \Leftrightarrow \quad x = \tanh y \quad \text{for } -1 \leq x \leq 1,\ y \in \Re$$

Graphs of Inverse Hyperbolic Functions

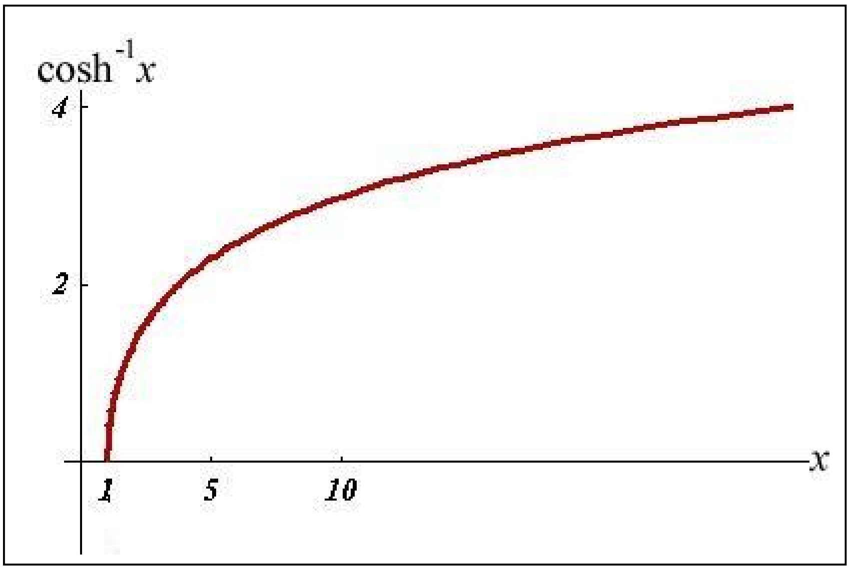
(i) $y = \sinh^{-1} x$

Domain: Range:



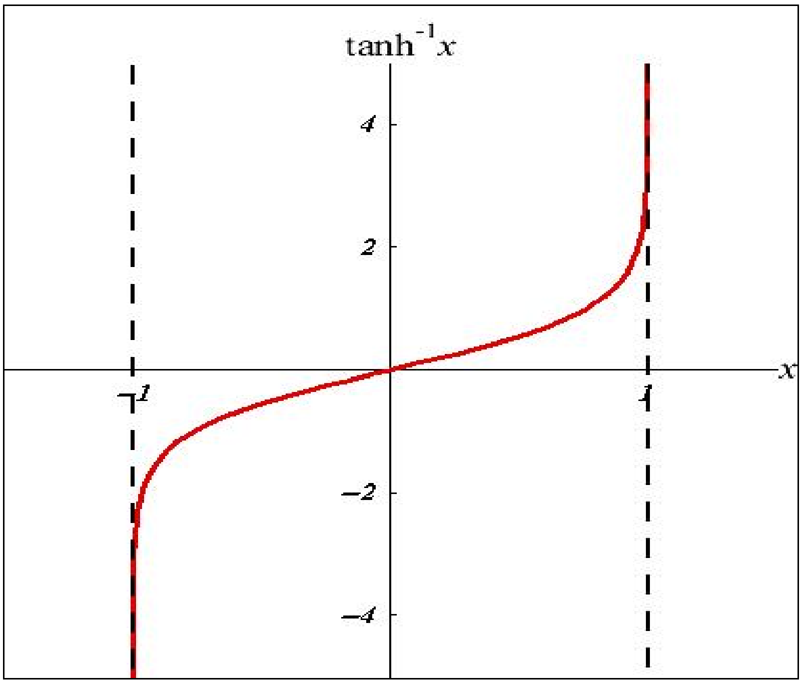
(ii) $y = \cosh^{-1} x$

Domain: Range:



(iii) $y = \tanh^{-1} x$

Domain: Range:



1.2.4 Log Form of the Inverse Hyperbolic Functions

It may be shown that

(a) $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$

(b) $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

(c) $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$

(d) $\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$

(e) $\sec h^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$

(f) $\cos ech^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{|x|}\right)$

Inverse Hyperbolic Cosine (Proof)

If we let $y = \cosh^{-1} x$, then

$$x = \cosh y = \frac{e^y + e^{-y}}{2}$$

Hence,

$$2x = e^y + e^{-y}$$

On rearrangement,

$$(e^y)^2 - 2xe^y + 1 = 0$$

Hence, (using formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$)

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

Since $e^y > 0$,

$$\therefore e^y = x + \sqrt{x^2 - 1}$$

Taking natural logarithms,

$$y = \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

Proof for $\sinh^{-1} x$

$y = \sinh^{-1} x$

$x = \sinh y = \dfrac{e^y - e^{-y}}{2}$

$\therefore 2x = e^y - e^{-y}$ (multiply with e^y)

$2xe^y = e^{2y} - 1$

$e^{2y} - 2xe^y - 1 = 0$

$e^y = x \pm \sqrt{x^2 + 1}$

Since $e^y > 0$,

$$\therefore e^y = x + \sqrt{x^2 + 1}$$

Taking natural logarithms,

$$y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

In the same way, we can find the expression for $\tanh^{-1} x$ in logarithmic form.

Examples 1.3: Evaluate

1) $\sinh^{-1}(0.5)$

2) $\cosh^{-1}(0.5)$

3) $\tanh^{-1}(-0.6)$