CHAPTER 1

Quantifiers & Proof Technique

QUANTIFIERS

- Most of the statements in mathematics and computer science are not described properly by the propositions.
- Since most of the statements in mathematics and computer science use **variables**, the system of logic must be extended to include **statements with the variables**.

QUANTIFIERS (cont.)

- Let $P(x)$ is a statement with variable x and A is a set.
- P a **propositional function** or also known as **predicate** if for each x in A, $P(x)$ is a proposition.
- Set A is the **domain of discourse** of P.
- Domain of discourse -> the particular domain of the variable in a propositional function.

QUANTIFIERS (cont.)

- A **predicate** is a statement that contains variables.

- **Example:**

  $P(x) : x > 3$

  $Q(x,y) : x = y + 3$

  $R(x,y,z) : x + y = z$

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Example

- $x^2 + 4x$ is an odd integer

  (domain of discourse is set of positive numbers).
- $x^2 - x - 6 = 0$

  (domain of discourse is set of real numbers).
- UTM is rated as Research University in Malaysia

  (domain of discourse is set of research university in Malaysia).

QUANTIFIERS (cont.)

- A predicate becomes a proposition if the variable(s) contained is(are)
  - **Assigned specific value(s)**
  - **Quantified**

Example

- $P(x) : x > 3.$

  What are the truth values of $P(4)$ and $P(2)$?

- $Q(x,y) : x = y + 3.$

  What are the truth values of $Q(1,2)$ and $Q(3,0)$?

QUANTIFIERS (cont.)

- Two types of quantifiers:
  - **Universal**
  - **Existential**

QUANTIFIERS (cont.)

- Let *A* be a propositional function with domain of discourse *B*. The statement

  for every x, $A(x)$

  is **universally quantified statement**

- Symbol ∀ called a **universal quantifier** is used "**for every**".
- Can be read as "**for all**", "**for any**".

QUANTIFIERS (cont.)

- The statement can be written as

$$\forall x\, A(x)$$

- Above statement is true if $A(x)$ is true for every x in B (false if $A(x)$ is false for at least one x in B).
- A value x in the domain of discourse that makes the statement $A(x)$ *false* is called a **counterexample** to the statement.

Example

- Let the universally quantified statement is

$$\forall x \, (x^2 \geq 0)$$

- Domain of discourse is the set of real numbers.

- **This statement is true** because for every real number x, it is true that the square of x is positive or zero.

Example

- Let the universally quantified statement is

$$\forall x\ (x^2 \leq 9)$$

- Domain of discourse is a set B = {1, 2, 3, 4}
- When x = 4, the statement produce false value.
- Thus, **the above statement is false** and the counterexample is 4.

QUANTIFIERS (cont.)

- Easy to prove a universally quantified statement is true or false if the domain of discourse is not too large.
- What happen if the domain of discourse contains a large number of elements?
- For example, a set of integer from 1 to 100, the set of positive integers, the set of real numbers or a set of students in Faculty of Computing. It will be hard to show that every element in the set is *true*.

Use existential quantifier!!

QUANTIFIERS (cont.)

- Let *A* be a propositional function with domain of discourse *B*. The statement

  There exist x, $A(x)$

  is **existentially quantified statement**
- Symbol ∃ called an **existential quantifier** is used "**there exist**".
- Can be read as "**for some**", "**for at least one**".

QUANTIFIERS (cont.)

- The statement can be written as

$$\exists x\, A(x)$$

- Above statement is true if $A(x)$ is true for at least one x in B (false if every x in B makes the statement $A(x)$ false).

- **Just find one x that makes $A(x)$ true!**

Example

- Let the existentially quantified statement is

$$\exists x\left(\frac{x}{x^2+1}=\frac{2}{5}\right)$$

- Domain of discourse is the set of real numbers.
- **Statement is true** because it is possible to find at least one real number x to make the proposition true.

- For example, if x = 2, we obtain the true proposition as below

$$\left(\frac{x}{x^2+1}=\frac{2}{5}\right)=\left(\frac{2}{2^2+1}=\frac{2}{5}\right)$$

Negation of Quantifiers

- Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.

$$\neg (\forall x\, P(x)) \;;\; \exists x \neg P(x)$$

$$\neg (\exists x\, P(x)) \;;\; \forall x \neg P(x)$$

Example

- Let $P(x)$ = x is taking Discrete Structure course with the domain of discourse is the set of all students.

  - $\forall x\, P(x)$: All students are taking Discrete Structure course.

  - $\exists x\, P(x)$: There is some students who are taking Discrete Structure course.

$$\neg\ (\exists x\, P(x))\ ;\ \forall x \neg P(x)$$

$\neg\exists x\, P(x)$: None of the students are taking Discrete Structure course.

$\forall x \neg P(x)$: All students are not taking Discrete Structure course.

$$\neg (\forall x\, P(x))\ ;\ \exists x \neg P(x)$$

$\neg\forall x\, P(x)$: Not all students are taking Discrete Structure course.

$\exists x \neg P(x)$: There is some students who are not taking Discrete Structure course

Proof Techniques

- **Mathematical systems consists:**
  - Axioms: assumed to be true.
  - Definitions: used to create new concepts.
  - Undefined terms: some terms that are not explicitly defined.
  - Theorem

Proof Techniques

- **Theorem**
  - Statement that can be shown to be true (under certain conditions)
  - Typically stated in one of three ways:
    - As Facts
    - As Implications
    - As Bi-implications

Proof Techniques (cont.)

Direct Proof (Direct Method)

- Proof of those theorems that can be expressed in the form $\forall x\ (P(x) \rightarrow Q(x))$, *D* is the domain of discourse.
- Select a particular, but arbitrarily chosen, member a of the domain *D.*
- Show that the statement $P(a) \rightarrow Q(a)$ is true. (Assume that $P(a)$ is true).
- Show that $Q(a)$ is true.
- By the rule of Universal Generalization (UG), $\forall x\ (\mathrm{P}(x) \rightarrow \mathrm{Q}(x))$ is true.

Example

For all integer x, if x is odd, then x^2 is odd

Or $P(x)$ = is an odd integer

$Q(x) = x^2$ is an odd integer

$$\forall x(P(x) \rightarrow Q(x))$$

the domain of discourse is set Z of all integer.

Can verify the theorem for certain value of x.

$x=3$, $x^2=9$; odd

Example (cont.)

- a is an odd integer

$$\Rightarrow a = 2n + 1$$ → *for some integer **n***

$$\Rightarrow a^2 = (2n+1)^2$$

$$\Rightarrow a^2 = 4n^2 + 4n + 1$$

$$\Rightarrow a^2 = 2(2n^2 + 2n) + 1$$

$$\Rightarrow a^2 = 2m + 1$$ → *where **m= 2n²** + 2n is an integer*

$$\Rightarrow a^2$$ → *is an odd integer*

Proof Techniques (cont.)

- **Indirect Proof**
  - The implication $p \rightarrow q$ is **equivalent** to the implication $(\neg q \rightarrow \neg p)$
  - Therefore, in order to show that $p \rightarrow q$ is true, one can also **show that the implication $(\neg q \rightarrow \neg p)$ is true**.
  - To show that $(\neg q \rightarrow \neg p)$ is true, assume that the negation of q is true and prove that the negation of p is true.

Example

P(n) : n^2+3 is an odd number

Q(n) : *n* is even number

$$\forall n(P(n) \rightarrow Q(n))$$

$$P(n) \rightarrow Q(n) \equiv \neg Q(n) \rightarrow \neg P(n)$$

- ¬ *Q(n)* is true , *n* is not even (*n* is odd), so *n=2k+1*

$$\begin{aligned} n^2 + 3 &= (2k+1)^2 + 3 \\ &= 4k^2 + 4k + 1 + 3 \\ &= 4k^2 + 4k + 4 \\ &= 2(2k^2 + 2k + 2) \end{aligned}$$

Example (cont.)

$$n^2 + 3 = (2k+1)^2 + 3$$
$$= 4k^2 + 4k + 1 + 3$$
$$= 4k^2 + 4k + 4$$
$$= 2(2k^2 + 2k + 2)$$

$$t = 2k^2 + 2k + 2$$
$$n^2 + 3 = 2t$$

→ *t is integer*

n^2+3 is an even integer, thus ¬ *P(n)* is true

Proof Techniques (cont.)

Proof by Contradiction

Assume that the hypothesis is true and that the conclusion is false and then, arrive at a contradiction.

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Example

Prove that there are infinitely many prime numbers.

Proof:

- Assume there are not infinitely many prime numbers, therefore they are can be listed, i.e. $p_1, p_2, \ldots, p_n$
- Consider the number $q = p_1 \times p_2 \times \ldots \times p_n + 1$.
- q is either prime or not divisible, but not listed above. Therefore, q is a prime. However, it was not listed.
- **Contradiction**! Therefore, there are infinitely many primes numbers.

Example

- For all real numbers x and y, if $x+y \geq 2$, then either $x \geq 1$ or $y \geq 1$.

Proof

- Suppose that the conclusion is false. Then

  $x < 1$ and $y < 1$

  Add these inequalities, $x+y < 1+1 = 2$ (**$x+y < 2$**)

- **Contradiction**
- Thus we conclude that the statement is true.