CHAPTER 2
Part 3

Recurrence Relation
&
Recursive Algorithms

Definition

- A **recurrence relation** is an equation that defines a sequence $\{a_n, a_{n+1}, \ldots\}$ based on a rule that gives the next term as a function of one or more of the previous term in the sequence for all integers n with $n \geq n_0$,

  $(a_0,\ a_1, \ldots \quad)$

- A sequence is called a **solution** of a recurrence relation if it terms satisfy the recurrence relation.

Simple Recurrence Relation

The simplest form of a **recurrence relation** is the case where the next term depends only on the immediately previous term.

$$3, 8, 13, 18, 23, \ldots$$

The above sequence shows a pattern, given initial condition, a_0 =3:

$$3,\ 3+5,\ 8+5,\ 13+5,\ 18+5, \ldots$$
$$a_0, a_1, a_2, a_3, a_4, a_5, \ldots$$

generated from an equation:

$$a_n = a_{n-1} + 5, \qquad n \geq 1$$

n_{th} term of a sequence

Recurrence relation can be used to compute any n-*th* term of the sequence, given the initial condition, a_0:

$$a_n = a_{n-1} + 5\ ,\quad n \geq 1$$

$$3, 8, 13, 18, 23, 28, a_{n-1} + 5, \ldots$$

$$a_1 = a_0 + 5\ ,\quad 3 + 5 = 8$$

$$a_2 = a_1 + 5\ ,\quad 8 + 5 = 13$$

$$a_3 = a_2 + 5\ ,\quad 13 + 5 = 18$$

$$\vdots$$

$$a_5 = a_4 + 5\ ,\quad 23 + 5 = 28$$

Example 1

Consider the following sequence:

$$3, 9, 27, 81, 243, \ldots$$

The above sequence shows a pattern:

$3^1,\ 3^2,\ 3^3,\ 3^4,\ 3^5, \ldots$ $\quad$ $3^1,\ 3^2,\ 3^3,\ 3^4,\ 3^5, \ldots$

$a_1,\ a_2,\ a_3,\ a_4,\ a_5, \ldots$ **or** $a_0, a_1, a_2,\ a_3,\ a_4,\ a_5, \ldots$

Recurrence relation is defined by:

$a_n = 3^n, n \geq 1$ **or** $a_n = 3^{n+1}, n \geq 0$

Example 2

Given initial condition, $a_0 = 1$ and recurrence relation:

$$\boldsymbol{a_n = 1 + 2a_{n-1} \ , n \geq 1}$$

First few sequence are:

$a_1 = 1 + 2\,(1) = 3$
$a_2 = 1 + 2(3) = 7$
$a_3 = 1 + 2(7) = 15$

1, 3, 7, 15, 31, 63, …

Example 3

Given initial condition, $a_0 = 1$, $a_1 = 2$ and recurrence relation:

$$\boldsymbol{a_n = 3(a_{n-1} + a_{n-2}),\ n \geq 2}$$

First few sequence are:

$$a_2 = 3(2 + 1) = 9$$
$$a_3 = 3(9 + 2) = 33$$
$$a_4 = 3(33 + 9) = 126$$

1, 2, 9, 33, 126, 477, 1809, 6858, 26001,…

Exercise

1) $a_n = 6a_{n-1} - 9a_{n-2}$, where $a_0 = 2$, $a_1 = 3$

2) $a_n = 2a_{n-1} - a_{n-2}$, where $a_0 = 5$, $a_1 = 3$

Example 4

For a photo shoot, the staff at a company have been arranged such that there are 10 people in the front row and each row has 7 more people than in the row in front of it. Find the recurrence relation and compute number of staff in the first 5 rows.

Solution

Notice that the difference between the number of people in successive rows is a constant amount.

This means that the n_{th} term of this sequence can be found using:

$$\boldsymbol{a_n} = \boldsymbol{a_{n-1}} + 7, \ \ n \geq 2 \text{ with } \boldsymbol{a_1} = \mathbf{10}$$

Solution

Number of staff in the first 5 rows:

$\boldsymbol{a_1 = 10}$,

$\boldsymbol{a_2 = a_1} + 7,\ \ 10 + 7 = 17$

$\boldsymbol{a_3 = a_2} + 7,\ \ 17 + 7 = 24$

$\boldsymbol{a_4 = a_3} + 7,\ \ 24 + 7 = 31$

$\boldsymbol{a_5 = a_4} + 7,\ \ 31 + 7 = 38$

10, 17, 24, 31, 38

Example 5

Find a recurrence relation and initial condition for

1, 5, 17, 53, 161, 485, …

Solution:

Look at the differences between terms:

4, 12, 36, 108, …

The difference in the sequence is growing by a factor of 3.

4x3, 12x3, 36x3

Solution

However the original sequence is not.

1(3)=**3**, 5(3)=**15**,17(3)=**51**, . . .

1, 5, 17, 53, 161, 485, …

It appears that we always end up with **2 less than** the next term. So, the recurrence relation is defined by:

$a_n = 3(a_{n-1}) + 2, \; n \geq 1$, **with initial condition,** $a_0 = 1$

$$a_1 = 3(a_0) + 2 = 3(1)+2 = 5$$
$$a_2 = 3(a_1) + 2 = 3(5)+2 = 17$$
$$\vdots$$

Example 6

A depositor deposits RM 10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years? Let P_n denote the amount in the account after n years.

Solution

Derive the following **recurrence relation**:

$$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$$

Where, P_n = Current balance and P_{n-1} = Previous year balance and 0.05 is the compounding interest.

Solution

Initial condition, P_0 = 10,000. Then,

$P_1 = 1.05P_0$
$P_2 = 1.05P_1 = (1.05)^2P_0$
$P_3 = 1.05P_2 = (1.05)^3P_0$
…
$\boldsymbol{P_n = 1.05P_{n-1} = (1.05)^nP_0}$,

now we can use this formula to calculate n_{th} term without iteration

Solution

Let us use this formula to find P_{30} under the initial condition $P_0 = 10,000$:

$P_{30} = (1.05)^{30}(10,000) = 43,219.42$

After 30 years, the account contains **RM 43,219.42**.

Example

Consider the following sequence:

1, 5, 9, 13, 17

Find the recurrence relation that defines the above sequence.

Answer:

$$a_n = a_{n-1} + 4 \quad , n \geq 1 \text{ with } a_0 = 1$$

Exercise

A basketball is dropped onto the ground from a height of 15 feet. On each bounce, the ball reaches a maximum height 55% of its previous maximum height.

a)Write a recursive formula, a_n , that completely defines the height reached on the n_{th} bounce, where the first term in the sequence is the height reached on the ball's first bounce.

b)How high does the basketball reach after the 4_{th} bounce? Give your answer to two decimal places.

Exercise

A grain elevator company receives 300 tons of corn per week from farmers once harvest starts. The elevator operators plan to ship out 40% of the corn on hand each week the harvest season begins.

i) If the company has 800 tons of corn on hand at the beginning of harvest, what reccurrence relation describes the amount of corn on hand at the end of each week throughout the harvest season?

ii) Using the reccurance relation obtain in (i), find the amount of corn on hand at the end of week 3.

Exercise

Determine s_5 if $s_0, s_1, s_2, ...$ is a sequence satisfying the given recurrence relation and initial conditions.

$$s_n = 2s_{n-1} + s_{n-2} - s_{n-3} \quad \text{for } n \geq 3,\ s_0 = 2,\ s_1 = -1,\ s_2 = 4$$

Exercise

A consumer purchased items costing RM280 with a department store credit card that charges 1.5% interest per month compounded monthly. Write a recurrence relation and initial condition for b_n, the balance of the consumer's account after n months if no further charges occur and the minimum monthly payment of RM25 is made.

Recursive Algorithms

- A recursive procedure is a procedure that invokes itself.
- A recursive algorithm is an algorithm that contains a recursive procedure.
- Recursion is a powerful, elegant and natural way to solve a large class of problems.

example

Factorial problem

- If $n \geq 1$,

$$n! = n\ (n - 1)\ ....2.\ 1$$

and $0! = 1$

- Notice that, if $n \geq 2$, n factorial can be written,

$$n! = n\ (n - 1)(n - 2)\ ....2.\ 1$$

$$= n.\ (n-1)!$$

example

- n=5

- 5!

$$5! = 5.\ 4.\ 3.\ 2.\ 1 = 5.\ 4!$$

$$4! = 4.\ 3!$$

$$3! = 3.\ 2!$$

Algorithm

- Input: n, integer ≥ 0
- Output: n!
- Factorial (n) {

    if (n=0)

        return 1

    return n*factorial(n-1)

  }

example

- Fibonacci sequence, f_n

$$f_1 = 1$$

$$f_2 = 1$$

$$f_n = f_{n-1} + f_{n-2}, \qquad n \geq 3$$

1, 1, 2, 3, 5, 8, 13, ...

Algorithm

- Input: *n*
- Output: $f(n)$
- $f(n)$ {

```
    if (n=1 or n=2)
        return 1
    return f(n-1) + f(n-2)
}
```

Example

Consider the following arithmetic sequence: 1, 3, 5, 7, 9,...

Suppose a_n is the term sequence. The generating rule is $a_n = a_{n-1} + 2, n > 1$. The relevant recursive algorithm can be written as

```
f (n)
{   if (n = 1 )
            return 1
    return f (n – 1) +2
}
```

Use the above recursive algorithm to trace n = 4

Solutions

Trace the output if n = 4

f(4)

n = 4
Because n≠1
return f(3) +2

f(4) =7

Return 5 +2

f(3)

n = 3
Because n≠1
return f(2) +2

f(3) =5

Return 3 +2

f(2)

n = 2
Because n≠1
return f(1) +2

f(2) =3

Return 1 +2

f(1)

n = 1
Because n =1
return 1

f(1) =1

Return 1

Answer = 7

exercise

- Given,

$$s_n = 3 + 6 + 9 + \ldots\ldots + 3n$$

$$s_1 = 3 \qquad s_n = s_{n-1} + 3n \quad \text{for } n \geq 2,$$

- Write a recursive algorithm to compute S_n, $n \geq 1$.

Exercise: Test 1 (2018/2019)

1. Suppose that the number of bacteria in a colony triples every hour.
   a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.

   b) If 10 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

2. Find a recurrence relation for the balance B(k) owed at the end of k months on a loan of RM9000 at rate of 8% if a payment of RM150 is made each month.
   [Hint: Express B(k) in terms of B(k-1); the monthly interest is (0.07/12) (B(k-1))]

3. Write a recursive algorithm for computing n^2 where n is a nonnegative integer using the fact that $(n+1)^2 = n^2 + 2n + 1$.

Exercise

Write a recursive algorithm to find the n term of sequence defined by $a_1 = 2$, $a_2 = 3$, and $a_n = a_{n-1}a_{n-2}$ for $n \geq 3$. Then, use the algorithm to trace a_7.