UTM
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INSPIRING CREATIVE AND INNOVATIVE MINDS

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# Chapter 2
# (Part 1)

# Relations

# Relations

- A (binary) relation $R$ from a set $X$ to a set $Y$ is a subset of the Cartesian product $X \times Y$.
- If $(x,y) \in R$, we write

$$x \, R \, y \quad (x \text{ is related to } y)$$

(Binary) relation from $X$ to $Y$, where $x \in X$, $y \in Y$, $(x,y) \in X \times Y$ and $R \subseteq X \times Y$

$$x \, R \, y \leftrightarrow (x,y) \in R$$

# example

- $A = \{ 1, 2, 3, 4 \}$ , $B = \{ p, q, r \}$
- $R = \{ (1, q), (2, r), (3, q), (4, p) \}$
- $R \subseteq A \times B$
- $R$ is the relation from $A$ to $B$

  $1 \; R \; q$ $\qquad$ $3 \; \not R \; p$

AxB

(1,p),(1,q),(1,r),
(2,p),(2,q),(2,r),
(3,p),(3,q),(3,r),
(4,p),(4,q),(4,r)

## example

$aRb \leftrightarrow a - b \in Z^{even}$

- *Finite set: A*= { 1, 2 } , *B*= { 1, 2, 3 }

  $R$ = { (1, 1), (2, 2), (1, 3) }

- *Infinite set: A=Z* and *B=Z*

  $R$ = { …(-3, -1), (-2, 2), (1, 3), …. }

(note: $Z$ is set of integers)

# example

- $A$ = { New Delhi, Ottawa, London, Paris, Washington }
- $B$ = { Canada, England, India, France, United States }
- Let $x \in A$, $y \in B$. Define the relation between $x$ and $y$ by "$x$ is the capital of $y$"

# example

- R = { (New Delhi, India),
  (Ottawa, Canada),
  (London, England),
  (Paris, France),
  (Washington, United States) }

# example

- "less than" relation from $A$={0, 1, 2} to $B$={1, 2, 3}
- Traditional notation:

  0<1, 0<2, 0<3, 1<2, 1<3, 2<3
- Set notation

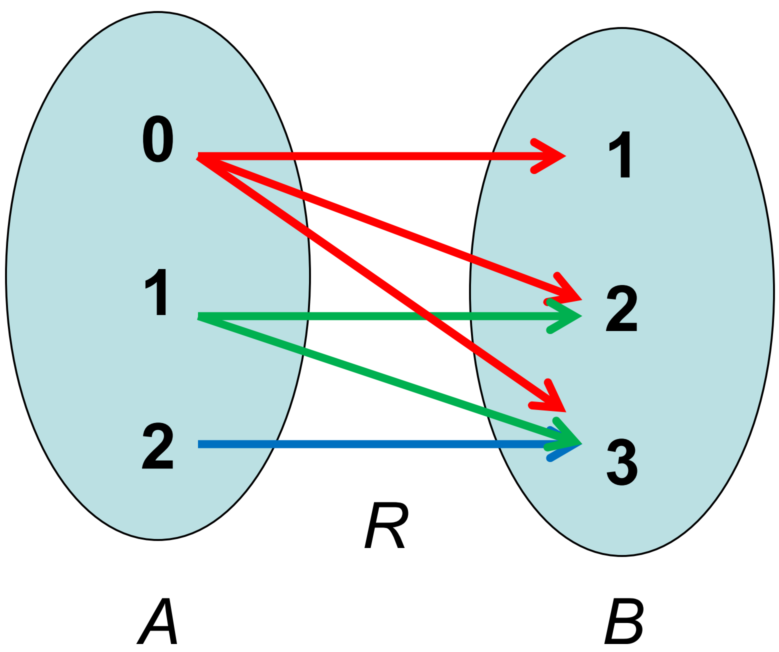
  $A \times B$ = { (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,1), (2,2), (2,3) }

  $R$ = { (0,1), (0,2), (0,3), (1,2), (1,3), (2,3) }

## example

$R = \{ (0,1), (0,2), (0,3), (1,2), (1,3), (2,3) \}$

- Arrow diagrams



# Domain and Range

- Let $R$, a relation from $A$ to $B$.
- The set, $\{ a \in A \mid (a,b) \in R \text{ for some } b \in B \}$ is called the **domain** of $R$.
- The set, $\{ b \in B \mid (a,b) \in R \text{ for some } a \in A \}$ is called the **range** of $R$.
- In case $A=B$, we call $R$ a(binary) **relation on** $A$.

## example

- Let $R$ be a relation on $X = \{1, 2, 3, 4\}$ defined by $(x,y) \in R$ if $x \leq y$, and $x, y \in X$.

- Then,
  $R = \{ (1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4) \}$

- The domain and range of $R$ are both equal to $X$.

# example

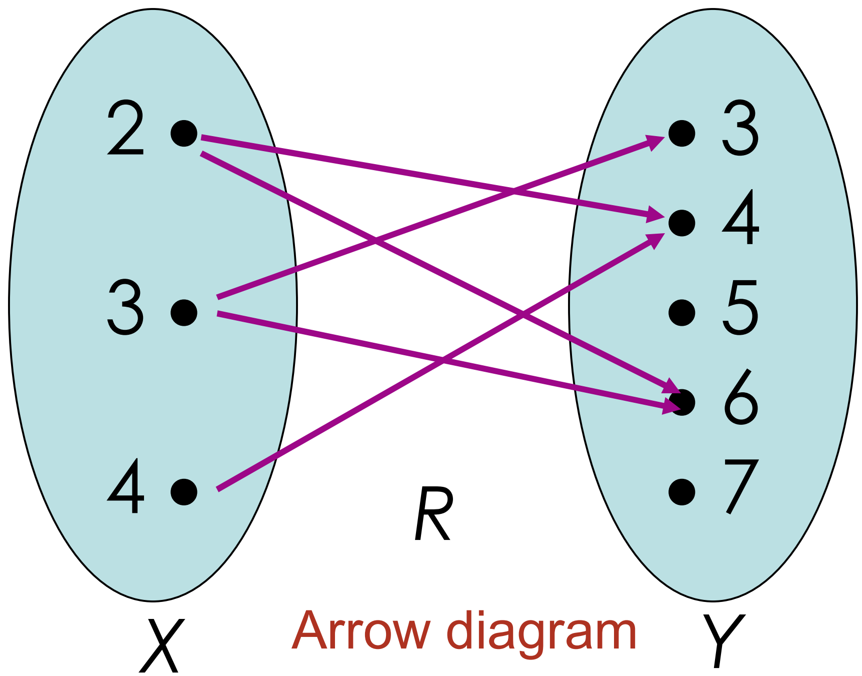
- Let $X = \{ 2, 3, 4 \}$ and $Y = \{ 3, 4, 5, 6, 7 \}$
  If we define a relation $R$ from $X$ to $Y$ by,
  $(x,y) \in R$ if $x$ divides $y$ (with zero remainder)

- We obtain,
  $R = \{ (2,4), (2,6), (3,3), (3,6), (4,4) \}$

  The domain of $R$ is $\{2,3,4\}$
  The range of $R$ is $\{3,4,6\}$

# example

$R$ = { (2,4), (2,6), (3,3), (3,6), (4,4) }




Arrow diagram

# Exercise

- Write the relation $R$ as $(x,y)\in R$

(a) The relation $R$ on {1, 2, 3, 4} defined by $(x,y)\in R$ if $x^2 \geq y$.

(b) The relation R on {1,2,3,4,5} defined by $(x,y)\in R$ if 3 divides x-y.

# Exercise

- Find range and domain for:

(a) The relation $R$ on {1, 2, 3, 4} defined by $(x,y) \in R$ if $x^2 \geq y$.

(b) The relation $R$ = { (1,2), (2,1), (3,3), (1,1), (2,2)} on $X$ = {1, 2, 3}

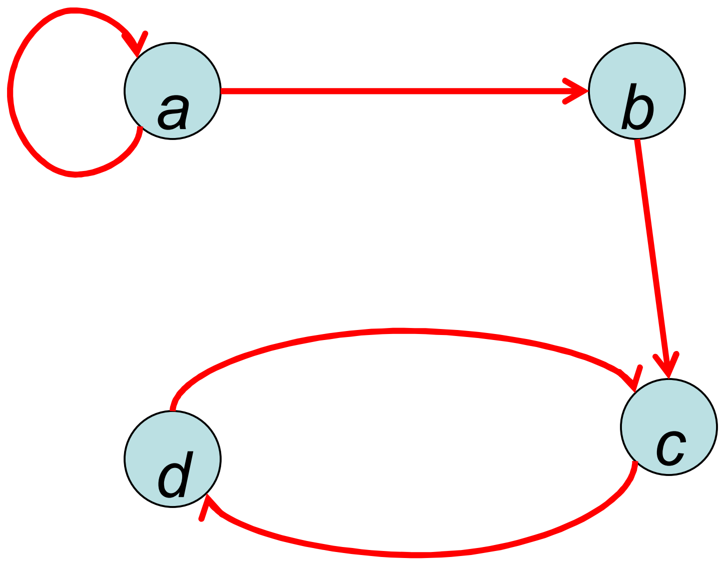
# Digraph

- An informative way to picture a relation on a set is to draw its digraph.
- Let *R* be a relation on a finite set *A*.
- Draw dots (vertices) to represent the elements of *A*.
- If the element $(a,b) \in R$, draw an arrow (called a directed edge) from a to b .

# example

- The relation $R$ on $A = \{a, b, c, d\}$,

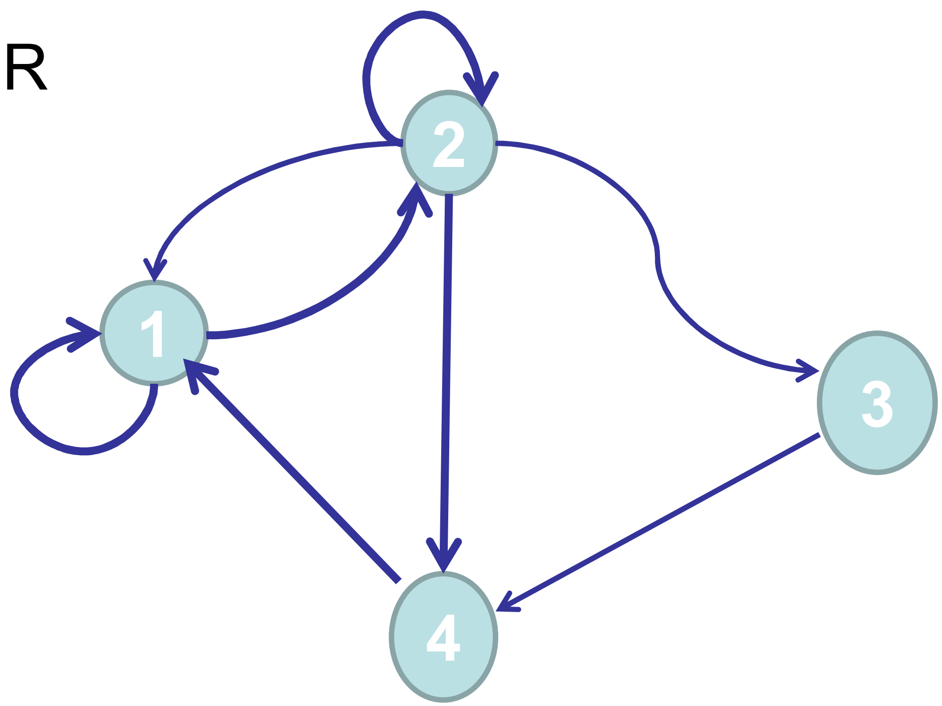
  $R = \{(a, a), (a, b), (c, d), (d, c), (b, c)\}$



# example

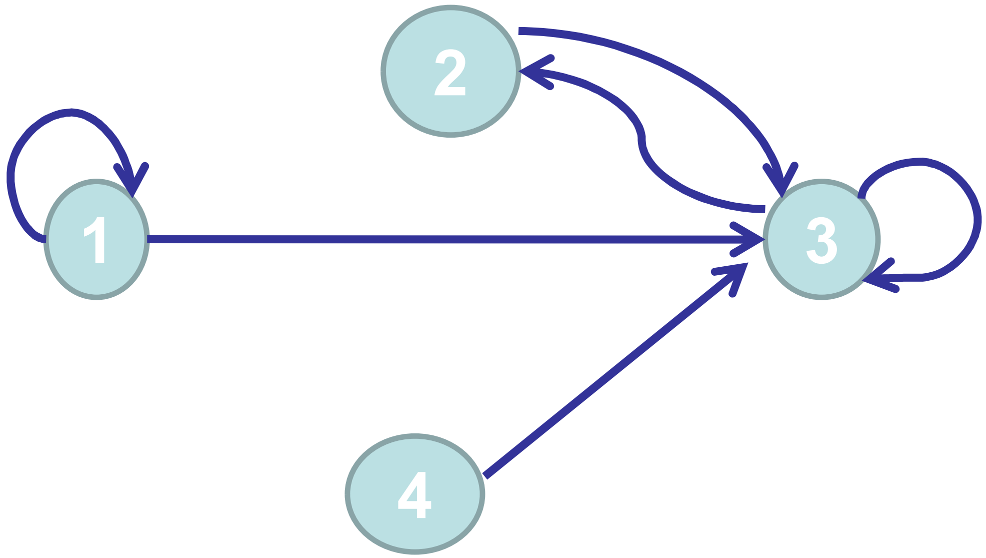
- Let, A = { 1,2,3,4} and R = { (1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4) , (4,1)}

- Draw the digraph of R



# example

Find the relation determined by digraph below.




- Since *a R b* if and only if there is an edge from *a* to *b*, so
  $R = \{ (1,1), (1,3), (2,3), (3,2), (3,3), (4,3) \}$

# exercise

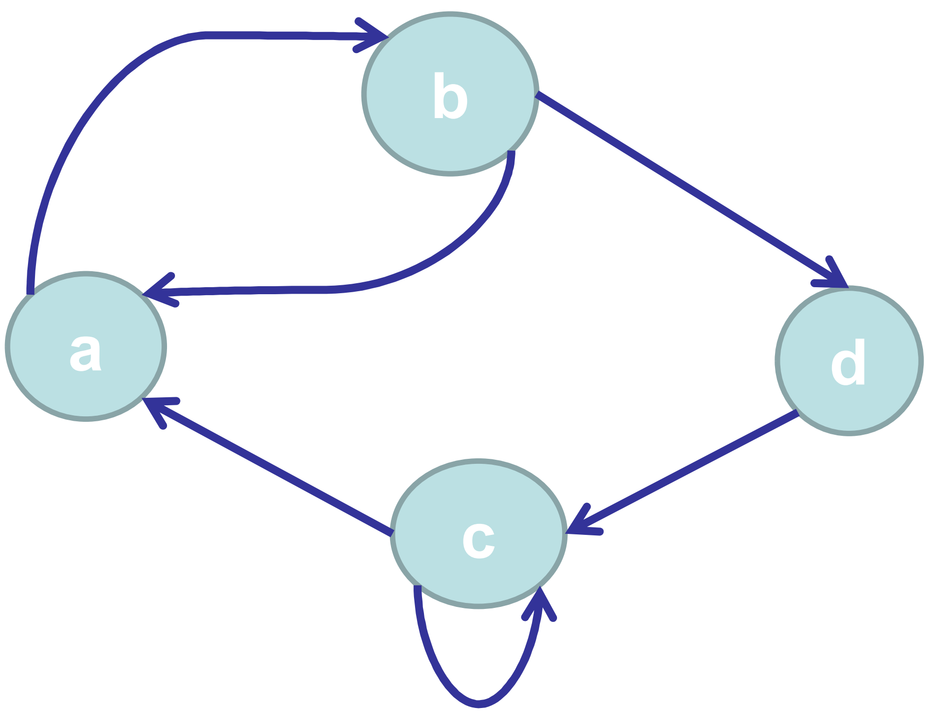
Draw the diagraph of the relation:

(a) $R$ = { (a,c), (b, d), (a,b), (c,d) }

(b) The relation $R$ on {1, 2, 3, 4} defined by $(x,y) \in R$ if $x^2 \geq y$

# exercise

Write the relation as a set of ordered pair.



# Matrices of Relations

- A matrix is a convenient way to represent a relation *R* from *A* to *B*.
- Label the rows with the elements of *A* (in some arbitrary order)
- Label the columns with the elements of *B* (in some arbitrary order)

# Matrices of Relations

- Let $A=\{a_1, a_2,\ldots, a_n\}$ and $B=\{b_1, b_2,\ldots, b_p\}$ be finite nonempty sets.
- Let $R$ be a relation from $A$ into $B$.
- Let $M_R = [m_{ij}]_{nxp}$ be the Boolean nxp matrix, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$

# Matrices of Relations

$$M_R = \begin{bmatrix} m_{11} & m_{12} & .. & .. & m_{1p} \\ m_{21} & m_{22} & .. & .. & m_{2p} \\ : & : & .. & .. & : \\ : & : & .. & .. & : \\ m_{n1} & m_{n2} & .. & .. & m_{np} \end{bmatrix}$$

## example

- Let $A = \{1, 3, 5\}$ and $B = \{ 1, 2\}$
- Let $R$ be a relation from $A$ to $B$ and $R = \{(1,1), (3,2), (5,1)\}$
- Then the matrix represent $R$ is

$$\begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

# example

- The relation,
  $R = \{ (1,b),(1,d),(2,c),(3,c),(3,b),(4,a) \}$
  from, $X = \{ 1, 2, 3, 4 \}$ to $Y = \{ a, b, c, d \}$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \textbf{or} \quad \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} d & b & a & c \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

# example

- The matrix of the relation $R$ from { 2, 3, 4 } to { 5, 6, 7, 8 } defined by

$$x \; R \; y \quad \text{if } x \text{ divides } y$$

$$\begin{array}{c c} & \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

# example

- Let A={ *a*, *b*, *c*, *d* }
- Let *R* be a relation on *A*.
- *R* = { (*a*,*a*),(*b*,*b*),(*c*,*c*),(*d*,*d*),(*b*,*c*),(*c*,*b*) }

$$
\begin{array}{c c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

# exercise

- Let $A=\{ 1, 2, 3, 4 \}$ and $R$ be a relation on $A$.
  $R = \{ (1,2),(1,3),(1,4),(2,3),(2,4),(3,4) \}$

(a) What is $R$ (represent)?

(b) What is matrix representation of $R$?

# In Degree and Out Degree

- If R is a relation on a set $A$ and $a \in A$, then the in-degree of $a$ (relative to relation $R$) is the number of $b \in A$ such that $(b, a) \in R$.

- The out degree of $a$ is the number of $b \in A$ such that $(a, b) \in R$

# In Degree and Out Degree

- Meaning that, in terms of the digraph of R, is that the in-degree of a vertex is

  "the number of edges terminating at the vertex"

- The out-degree of a vertex is

  " the number of edges leaving the vertex"

# Example

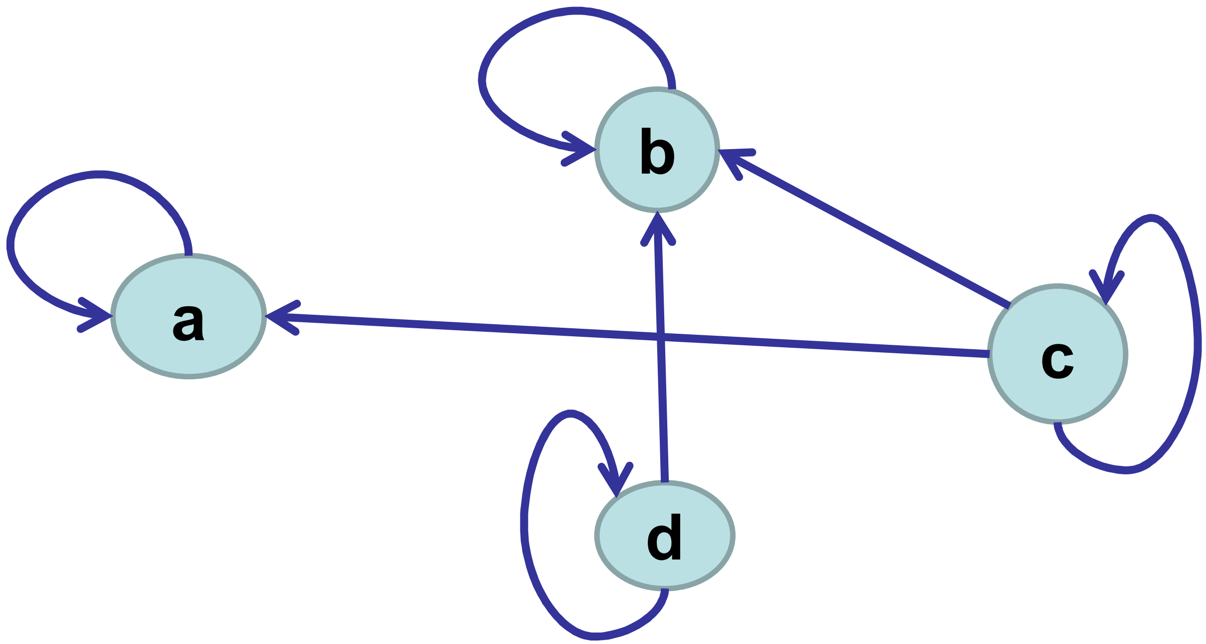
- Let $A = \{a, b, c, d\}$, and let $R$ be the relation on $A$ that has the matrix (given below)

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

- Construct the digraph of $R$, and list in-degrees and out-degrees of all vertices.

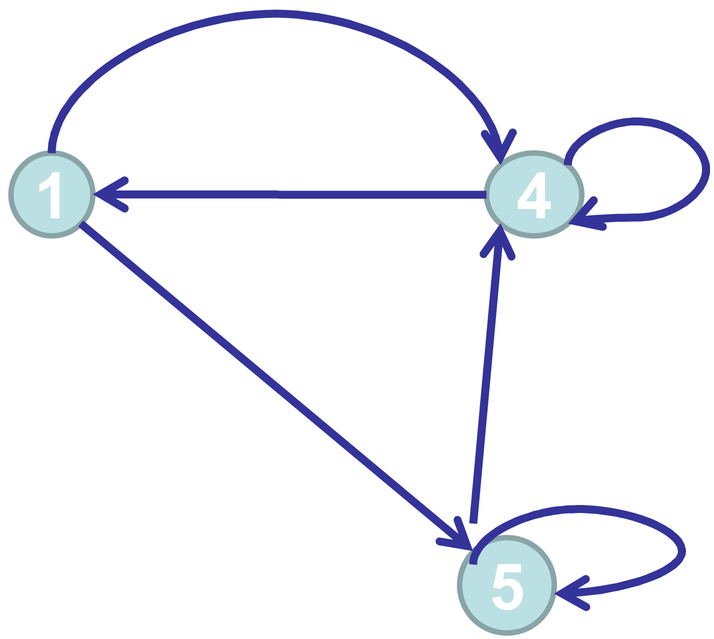
# Example

| | a | b | c | d |
|---|---|---|---|---|
| In-degree | 2 | 3 | 1 | 1 |
| Out-degree | 1 | 1 | 3 | 2 |



# exercise

- Let $A$ = {1, 4, 5} and let $R$ be given by the digraph shown below.
- Find $M_R$ and $R$



# example

- An airline services the five cities $c_1$, $c_2$, $c_3$, $c_4$ and $c_5$.
- Table below gives the cost (in dollars) of going from $c_i$ to $c_j$. Thus the cost of going from $c_1$ to $c_3$ is \$100, while the cost of going from $c_4$ to $c_2$ is \$200

| To<br>from | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ |
|---|---|---|---|---|---|
| $c_1$ | | 140 | 100 | 150 | 200 |
| $c_2$ | 190 | | 200 | 160 | 220 |
| $c_3$ | 110 | 180 | | 190 | 250 |
| $c_4$ | 190 | 200 | 120 | | 150 |
| $c_5$ | 200 | 100 | 200 | 150 | |

## example

- If the relation $R$ on the set of cities $A = \{c_1, c_2, c_3, c_4, c_5\}$ : $c_i \; R \; c_j$ if and only if the cost of going from $c_i$ to $c_j$ is defined and less than or equal to \$180.

- Find $R$.

$$R = \{(c_1,c_2), (c_1,c_3), (c_1,c_4), (c_2,c_4),(c_3, c_1), (c_3,c_2), (c_4,c_3), (c_4,c_5), (c_4,c_5), (c_5,c_2), (c_5,c_4)\}$$

# exercise

- From the previous example, find the matrices of relations for *R.*

$$R = \{(c_1,c_2), (c_1,c_3), (c_1,c_4), (c_2,c_4),(c_3, c_1), (c_3,c_2), (c_4,c_3), (c_4,c_5), (c_4,c_5), (c_5,c_2), (c_5,c_4)\}$$

# Reflexive Relations

- A relation $R$ on a set $X$ is called reflexive if $(x,x) \in R$ for every $x \in X$.

- That is, if $xRx$ for all $x \in X$.

($R$ is reflexive if every element $x \in X$ is related to itself)

# example

- The relation $R$ on $X = \{1, 2, 3, 4\}$ defined by $(x,y) \in R$ if $x \leq y$, $x,y \in X$ is reflexive because for each element $x \in X$, $(x,x) \in R$

- (1,1), (2,2), (3,3), (4,4) are each in $R$.

# example

- The relation,

  $R = \{ (a,a), (b,c), (c,b), (d,d) \}$

  on $X=\{a, b, c, d\}$

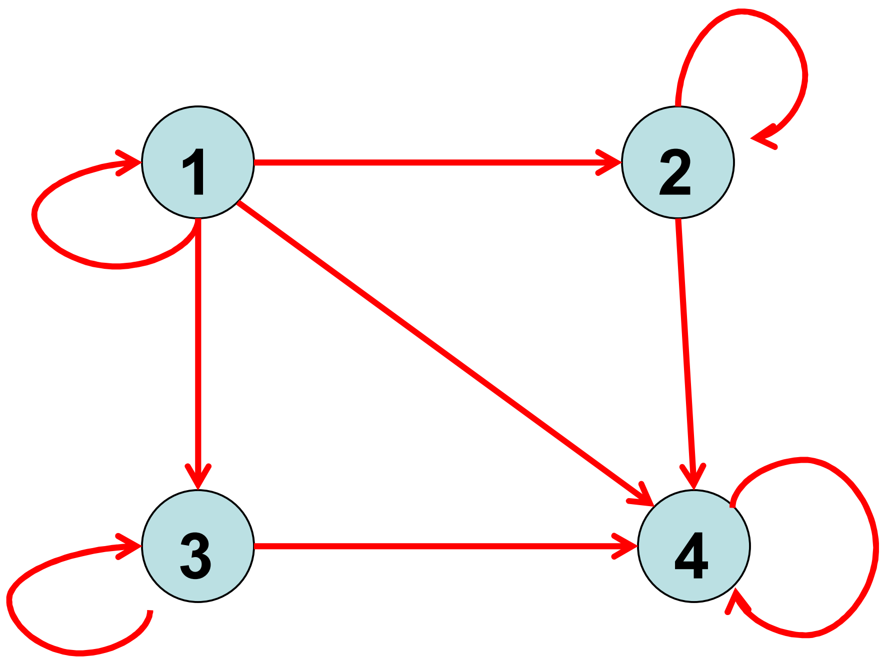
  is not reflexive.

- For example, $b \in X$, but $(b,b) \notin R$

# Reflexive Relations

- The digraph of a reflexive relation has a loop at every vertex.

- example



# Reflexive Relations

## Irreflexive

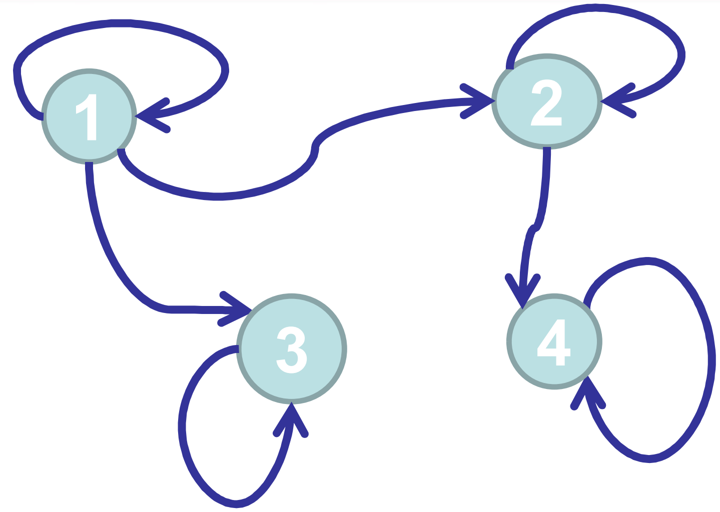
- A relation *R* on a set *A* is **irreflexive** if $x \not R x$ or $(x,x) \notin R;\ \forall x : x \in X$

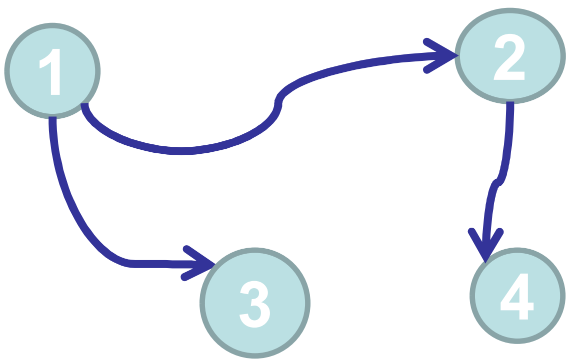
## Not Reflexive

- A relation *R* is **not reflexive** if at least one pair of $(x,x) \notin R,\ \forall x : x \in X$

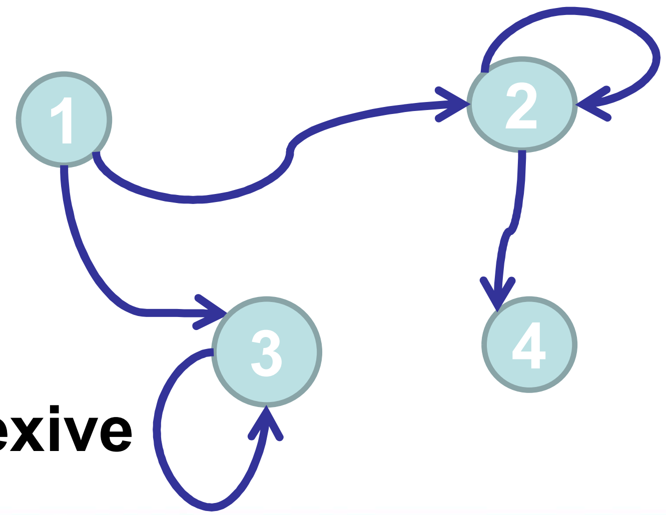
# example

Reflexive




Irreflexive




Not reflexive

# example

- Consider the following relations on the set {1, 2, 3}

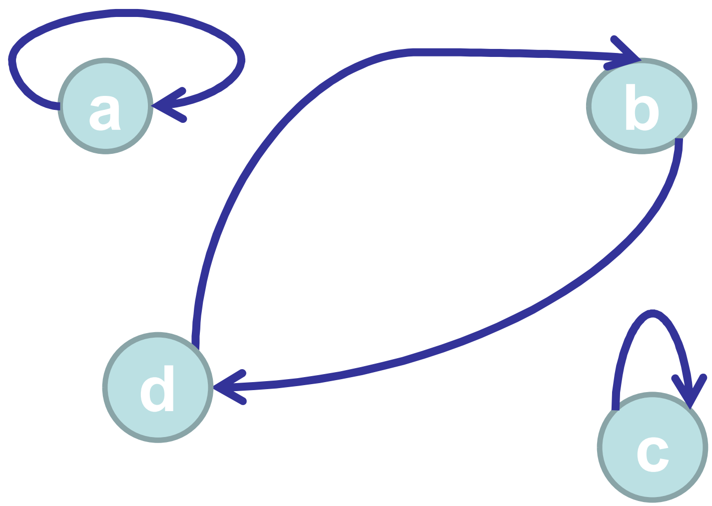
$R_1$ = { (1,1), (1,2), (2,1), (2,2), (3,1), (3,3) }
$R_2$ = { (1,1), (1,3), (2,2), (3,1) }
$R_3$ = { (2,3) }
$R_4$ = { (1,1) }

Which of them are reflexive?

# exercise

(i) Let $R$ be the relation on $X=\{1,2,3,4\}$ defined by $(x,y)\in R$ if $x \le y,\ x,y \in X$. Determine whether $R$ is a reflexive relation.

(ii) The relation $R$ on $X=\{a,b,c,d\}$ given by the below diagraph. Is $R$ a reflexive relation?



# Reflexive Relations

- The relation *R* is reflexive if and only if the matrix of relation has 1's on the main diagonal.

- example

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

# Reflexive Relations

The relation R is *irreflexive* if and only if the matrix relation have all 0's on its main diagonal

- example

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

# example

- The relation $R$ is **not reflexive**.

$$\begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$b \in X$
$(b,b) \notin R$

## exercise

Let A = { 1,2,3,4). Construct the matrix of relation of *R*. Then, determine whether the relation is reflexive, not reflexive or irreflexive.

(i) *R* = { (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4))

(ii) *R* = { (1,3), (1,1), (3,1), (1,2), (3,3), (4,4)}

(iii) *R* ={ (1,2), (1,3), (3,1), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4)}

(iv) *R* ={ (1,2), (1,3), (3,2), (1,4), (4,2), (3,4)}

# Symmetric Relations

- A relation $R$ on a set $X$ is called symmetric if for all $x, y \in X$, if $(x,y) \in R$, then $(y,x) \in R$.

$$\forall x,y \in X,\ (x,y) \in R \to (y,x) \in R$$

- Let $M$ be the matrix of relation $R$.

  The relation $R$ is symmetric if and only if for all $i$ and $j$, the $ij$th entry of $M$ is equal to the $ji$th entry of $M$.

# Symmetric Relations

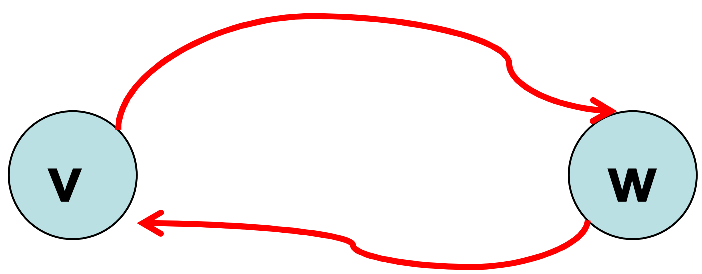
- The matrix of relation $M_R$ is symmetric if $M_R = M_R^T$

- Example

$$M_R = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} = M_R^T$$

# Symmetric Relations

- The digraph of a symmetric relation has the property that whenever there is a directed edge from v to w, there is also a directed edge from w to v.



## example

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$

$$\boxed{\begin{array}{l} (b,c) \in R \\ (c,b) \in R \end{array}}$$

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

symmetric

# example

- The relation $R$ on $X = \{ 1, 2, 3, 4 \}$, defined by

  $(x,y) \in R$ if $x \leq y,\ x,y \in X$

$$
\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
1 & 1 & 1 & 1 & 1 \\
2 & 0 & 1 & 1 & 1 \\
3 & 0 & 0 & 1 & 1 \\
4 & 0 & 0 & 0 & 1
\end{array}
$$

$(2,3) \in R$
$(3,2) \notin R$

not symmetric

# Antisymmetric Relations

- A relation, $R$ on a set $X$ is called antisymmetric, if for all $x,y \in X$, if $(x,y) \in R$ and $x \neq y$, then $(y,x) \notin R$.

- A relation $R$ on set $X$ is antisymmetric if $x \neq y$, whenever $xRy$, then $y \not R x$. In other word if whenever $xRy$, then $yRx$ then it implies that $x=y$

$$\forall x,y \in A, (x,y) \in R \wedge x \neq y \rightarrow (y, x) \notin R$$

Or

$$\forall x,y \in A, \ (x,y) \in R \wedge (y, x) \in R \rightarrow \ x = y$$

# Antisymmetric Relations

- Matrix $M_R = [M_{ij}]$ of an antisymmetric relation $R$ satisfies the property that if $i \neq j$, then $m_{ij}=0$ or $m_{ji}=0$.

- If $R$ is antisymmetric relation, then for different vertices $i$ and $j$ there cannot be an edge from vertex $i$ to vertex $j$ and an edge from vertex $j$ to vertex $I$

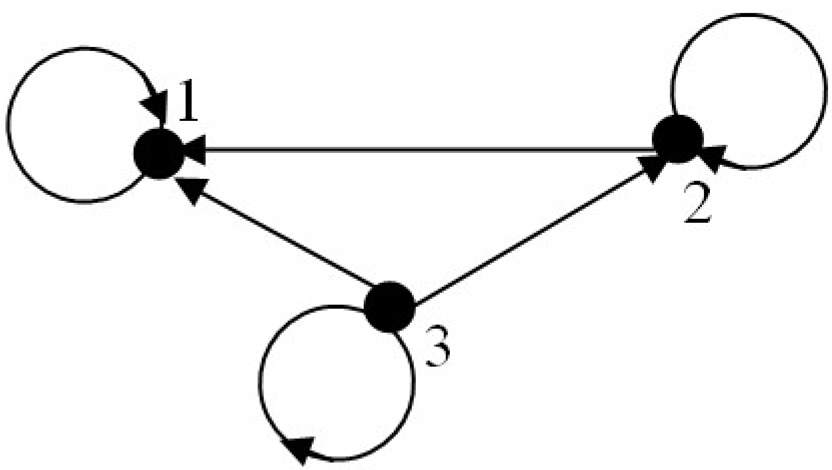
- At least one directed relation and one way

# Example

- Let $R$ be a relation on $A = \{1, 2, 3\}$ defined as $(a, b) \in R$ if $a \geq b$, $a, b \in A$ is an antisymmetric relation because for all $a, b \in A$, $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$, for example

$$(3, 2) \in R \text{ but } (2, 3) \notin R$$

$$(3, 3) \in R \text{ and } (3, 3) \in R \text{ implies } a = b$$



## example

- The relation *R* on *X* = { 1, 2, 3 ,4 } defined by,

  $(x,y) \in R$ if $x \leq y,\ x,y \in X$

$$\begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$(1,2) \in R$

$(2,1) \notin R$

antisymmetric

# example

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$

$$(b,c) \in R$$
$$(c,b) \in R$$

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

not antisymmetric

# example

- The relation
  $R = \{ (a,a), (b,b), (c,c) \}$
  on $X = \{ a, b, c \}$

- $R$ has no members of the form $(x,y)$ *with* $x \neq y$, then R is antisymmetric

$$
\begin{array}{cc}
 & \begin{array}{ccc} a & b & c \end{array} \\
\begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

# example

- Antisymmetric
- Reflexive
- Symmetric

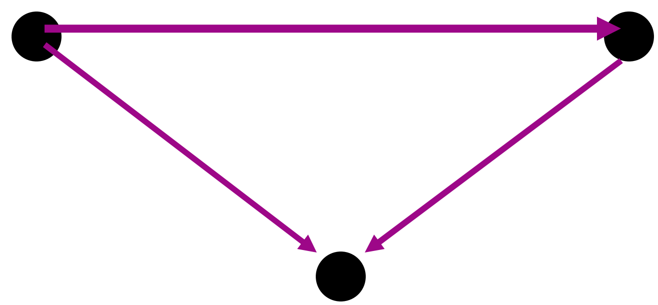
$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{array}$$

- "Antisymmetric" is not the same as "not symmetric"

# Antisymmetric Relations

- The digraph of an antisymmetric relation has at most one directed edge between each pair of vertices.

- Example



# Asymmetric

A relation $R$ on set $A$ is asymmetric if whenever $aRb$, then $b\not{R}a$.

$$\forall x,y \in X,\ (x,y) \in R \rightarrow (y,x) \notin R$$

In this sense, a relation is asymmetric if and only if it is both antisymmetric and irreflexive.

# Asymmetric

- The matrix $M_{R} = [m_{ij}]$ of an asymmetric relation *R* satisfies the property that

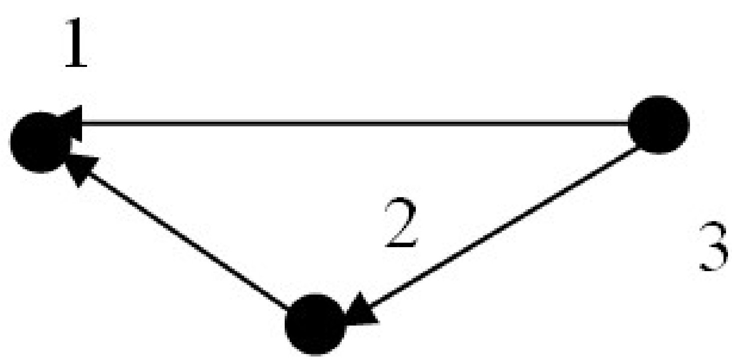
  If $m_{ij} = 1$ then $m_{ji} = 0$

  $m_{ii} = 0$ for all *i* (the main diagonal of matrix $M_R$ consists entirely of 0's or otherwise)
- If *R* is asymmetric relation, then the digraph of *R* cannot simultaneously have an edge from vertex *i* to vertex *j* and an edge from vertex *j* to vertex *i*
- All edges are "one way street"

# Example

- Let $R$ be the relation on A = {1, 2, 3} defined by $(a, b) \in R$ if $a > b$, $a,b \in A$ is an asymmetric relation because,

$$(2, 1) \in R \text{ but } (1, 2) \notin R$$
$$(3, 1) \in R \text{ but } (1, 3) \notin R$$
$$(3, 2) \in R \text{ but } (2, 3) \notin R$$



# Not Symmetric

- Let $R$ be a relation on a set $A$.
- Then $R$ is called ***not symmetric***, if for all $a$, $b \in A$, if $(a, b) \in R$, there exist $(b, a) \notin R$.

$$\exists a,b \in A, (a, b) \in R \rightarrow (b, a) \notin R$$

# Not Symmetric and Not Antisymmetric

- Let $R$ be a relation on a set $A$. Then $R$ is called ***not symmetric*** and ***not antisymmetric***, if for all $a$, $b \in A$, if $(a, b) \in R$, there exist $(b, a) \notin R$ and if $(a, b) \in R$, there exist $(b, a) \in R$.

$$\exists a,b \in A, (a, b) \in R \rightarrow (b, a) \in R$$

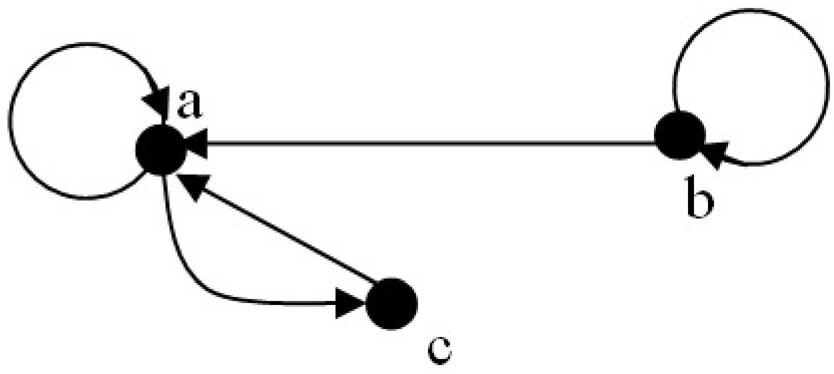
AND

$$\exists a,b \in A, (a, b) \in R \rightarrow (b, a) \notin R$$

# Example

- Relation $R = \{(a, c), (b, b), (c, a), (b, a), (a, a)\}$ on $A = \{a, b, c\}$ is not symmetric and not antisymmetric relation because there is,

$$(a,c), (c,a) \in R \text{ and also } (b,a) \in R \text{ but } (a,b) \notin R$$



# Example

1. Let $A=Z$, the set of integers and let $R=\{(a,b)\in A\times A \mid a<b\}$. So that $R$ is the relation "less than".

   Is $R$ symmetric, asymmetric or antisymmetric?

2. Let $A=\{1,2,3,4\}$ and let $R = \{(1,2), (2,2), (3,4), (4,1)\}$

   Determine whether $R$ symmetric, asymmetric or antisymmetric.

# Example

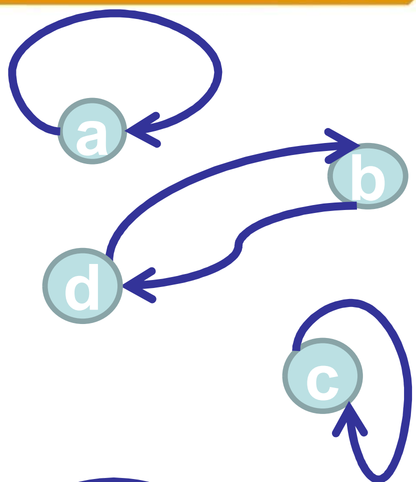
**Question 1**

- **Symmetric : If $a<b$, then it is not true that $b<a$, so $R$ is not symmetric**
- **Assymetric : If $a<b$ then $b>a$ ($b$ is greater than $a$), so $R$ is assymetric**
- **Antisymmetric : If $a \neq b$, then either $a>b$ or $b>a$, so $R$ is antisymmetric**

**Question 2**

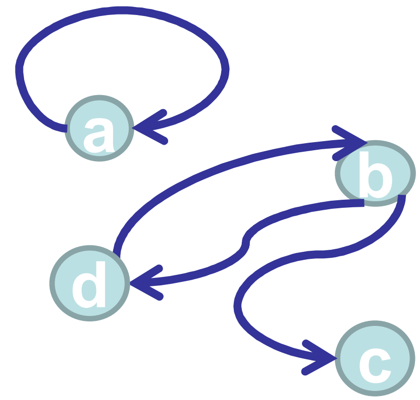
- **$R$ is not symmetric since $(1,2) \in R$, but $(2,1) \notin R$**
- **$R$ is not asymmetric , since $(2,2) \in R$**
- **$R$ is antisymmetric, since $a \neq b$, either $(a,b) \notin R$ or $(b,a) \notin R$**

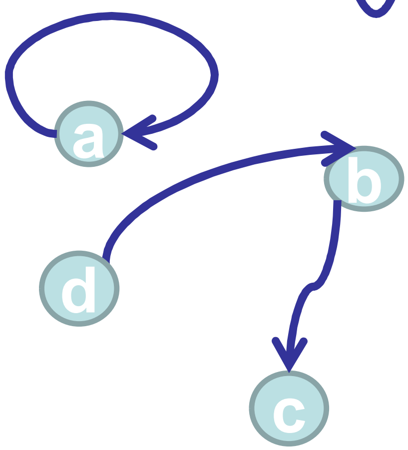
# Summary on Symmetric



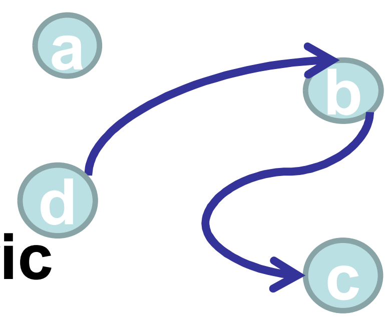

**Symmetric**

**Not Symmetric**




**Antisymmetric**




**Asymmetric**

# Exercise

Let A = { 1,2,3,4). Construct the matrix of relation of *R*. Then, determine whether the relation is symmetric, asymmetric, not symmetric or antisymmetric.

(i) *R* = { (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4))

(ii) *R* = { (1,3), (1,1), (3,1), (1,2), (3,3), (4,4)}

(iii) *R* ={ (1,2), (1,3), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4)}

# Transitive Relations

- A relation $R$ on a set $X$ is called transitive if for all $x,y,z \in X$,

  if $(x,y)$ and $(y,z) \in R$ then $(x,z) \in R$

- It is often convenient to say what it means for a relation to be not transitive.

- A relation $R$ on $X$ is **not transitive** if there exists $x$, $y$, and $z$ in X so that $xRy$ and $yRz$, but $x \not R z$. If such x,y, and z do not exist, then R is transitive.

# Transitive Relations

- Let $M_R$ be the matrix of relation R.
  The relation $R$ is transitive if,

$$M_R \otimes M_R = M_R.$$

$\otimes$ Boolean product.

# Boolean Algebra

| + | 1 | 0 |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 1 | 0 |

| . | 1 | 0 |
|---|---|---|
| 1 | 1 | 0 |
| 0 | 0 | 0 |

# example

- The relation $R$ on $X = \{ 1, 2, 3, 4 \}$ defined by

  $(x,y) \in R \qquad \text{if } x \le y,\ x,y \in X$

$$
\begin{array}{c|c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

# example

- $M_R \otimes M_R = M_R$

transitive

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# example

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$ is not transitive.
- $(b,c)$ and $(c,b) \in R$, but $(b,b) \notin R$.

# example

- Consider the following relations on the set {1, 2, 3}

R1 = { (1,1), (1,2), (2,3) }
R2 = { (1,2), (2,3), (1,3) }

- Which of them is transitive?

## example

Let $R$ be a relation on $A=\{1,2,3\}$ is defined by $(a,b)\in R$ if $a\leq b,\ a,b\in A$. Find $R$. Is $R$ a transitive relation?

Solution:

$R=\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

$R$ is a transitive relation because

$(1,2)$ and $(2,2)\in R$, $(1,2)\in R$

$(1,2)$ and $(2,3)\in R$, $(1,3)\in R$

$(1,3)$ and $(3,3)\in R$, $(1,3)\in R$

$(2,2)$ and $(2,3)\in R$, $(2,3)\in R$

$(2,3)$ and $(3,3)\in R$, $(2,3)\in R$

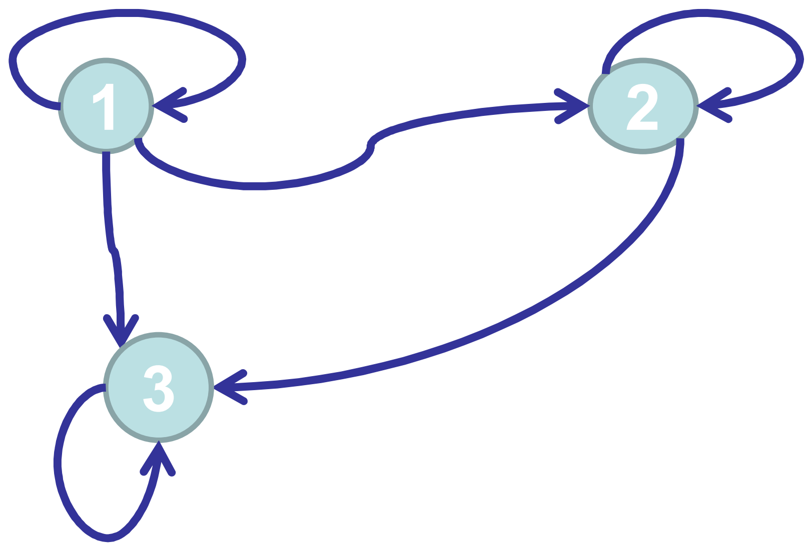
## Transitive Relations

In the digraph of *R*, *R* is a transitive relation if and only if there is a directed edge from one vertex *a* to another vertex *b*, an if there exits a directed edge from vertex *b* to vertex *c*, then there must exists a directed edge from *a* to *c*

# example

$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

The diagraph:



# example

The relation $R$ on $A=\{1,2,3\}$ defined by $(a,b)\in R$ if $a\leq b$, $a,b\in A$, is a transitive. The matrix of relation $M_R$,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

The product of boolean,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that,(1,2) and $(2,3)\in R$, $(1,3)\in R$

## example

The relation *R* on *A*={1,2,3} defined by *(a,b)*$\in$*R* if $a \le b$, *a,b*$\in$*A*, is a transitive. The matrix of relation $M_R$,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

The product of boolean,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that,(1,2) and (2,3)$\in$*R*, (1,3)$\in$*R*

# Example

The relation $R$ on $A$={a,b,c,d} IS r={(a,a), (b,b), (c,c), (d,d), (a,c), (c,b)} is not transitive. The matrix of relation $M_{R,}$

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$M_R \otimes M_R \neq M_R$$

The product of boolean,

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \otimes \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Note that,(a,c) and (c,b)$\in R$, (a,b)$\notin R$

# Equivalence Relations

- A relation *R* that is reflexive, symmetric and transitive on a set *X* is called an equivalence relation on *X*.

# example

Let $R$={(1,1), (1,3), (2,2), (3,1), (3,3)} on {1,2,3}, the matrix of the relation $M_R$,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

All the main diagonal matrix elements are 1 and the matrix is reflexive

# example

Let $R$={(1,1), (1,3), (2,2), (3,1), (3,3)} on {1,2,3}, the matrix of the relation $M_R$,

$$M_R = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

All the main diagonal matrix elements are 1 and the matrix is reflexive

## example

The transpose matrix $M_R$, $M_R^T$ is equal to $M_R$, so $R$ is symmetric

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \qquad M_R^T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

The product of Boolean show that the matrix is transitive.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

# example

- The relation, $R$={ (1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5) } on { 1,2,3,4,5 }

  - Reflexive?
  - Symmetric?
  - Transitive?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

# example

- Reflexive?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Reflexive √

# example

- Symmetric?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Symmetric √

# example

- Transitive?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

# example

- Reflexive
- Symmetric
- Transitive

$\left.\begin{array}{l}\text{Reflexive}\\ \text{Symmetric}\\ \text{Transitive}\end{array}\right\}$ Equivalence relation

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

# example

- The relation $R$ on $X$={ 1, 2, 3, 4 }, defined by
  $(x,y) \in R$ if $x \leq y,\ x,y \in X$

- Not symmetric
  - $(2,3) \in R$ but $(3,2) \notin R$

- $R$ is not equivalence relation on $X$.

# Partial Orders

- A relation, *R* on a set *X* is called a partial order if *R* is reflexive, antisymmetric, and transitive.

# example

Let $R$ be a relation on a set $A=\{1,2,3\}$ defined by $(a,b)\in R$ if $a\leq b,\ a,b\in R$.

$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

$R$ is reflexive, antisymmetric and transitive.

So $R$ is a partial order relation.

# example

- The relation *R* defined on the positive integers by

  $(x,y) \in R$ if $x$ divides $y$ (evenly)

  is reflexive, antisymmetric and transitive

- R is a partial order.

# example

- The relation *R* defined on the set of integers by

  $(x,y) \in R \quad \text{if } x \leq y$

  is reflexive, antisymmetric, and transitive.

- R is a partial order.

# exercise

The relation $R$ on the set {1,2,3,4,5} defined by the rule $(x,y) \in R$ if $x+y \leq 6$

(i) List the elements of R
(ii) Find the domain of R
(iii) Find the range of R
(iv) Is the relation of R refelxive, symmetric, assymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?

## exercise

The relation *R* on the set {1,2,3,4,5} defined by the rule $(x,y) \in R$ if 3 divides $x-y$

(i) List the elements of R

(ii) Find the domain of R

(iii) Find the range of R

(iv) Is the relation of R refelxive, symmetric, assymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?

## exercise

The relation *R* on the set {1,2,3,4,5} defined by the rule $(x,y) \in R$ if $x=y-1$

(i) List the elements of R
(ii) Find the domain of R
(iii) Find the range of R
(iv) Is the relation of R refelxive, symmetric, assymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?