CHAPTER 4
(Part 1)
Overview of Probability

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Probability

(Recall from subject SCSI1013 – Discrete Maths: Chapter 3 (Part 4 – probability theory)

- Probability – the systematic study of uncertainty.
- Example of activities involve with uncertainty:

Rolling a die

Tossing a coin

Select cards from deck

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Chance Experiments

Suppose:

- two six-sided die is rolled and they both land on sixes.
- a coin is flipped and it lands on heads.

These would be examples of **chance experiments.**

A **chance experiment** is any activity or situation in which there is uncertainty about which of two or more plausible outcomes will result.

Sample Space

Suppose a six-sided die is rolled. The possible outcomes are that the die could land with 1 dot up or 2, 3, 4, 5, or 6 dots up.

$$S = \{1, 2, 3, 4, 5, 6\}$$

This would be an example of a sample space (S).

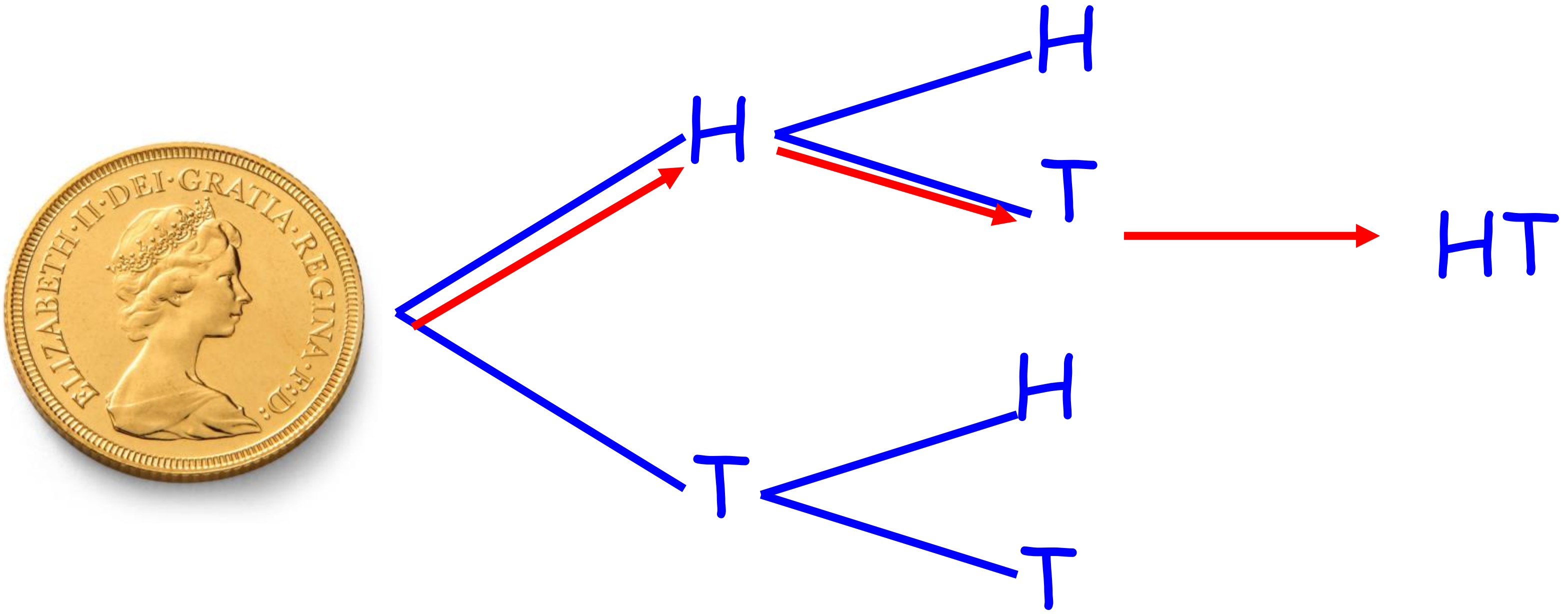
The collection of all possible outcomes of a chance experiment is the **sample space** for the experiment.

Sample Space (cont.)

Suppose two coins are flipped. The sample space would be:

S = {HH, HT, TH, TT} ; H = heads and T = tails

We can also use a tree diagram to represent a sample space.



Event

- Example: The car purchase chance experiment has a sample space:

  S = {MH, FH, MT, FT} ;

where M = male, F=female, H=hybrid, T= traditional.

- We might focus on a group of outcomes that involve the purchase of a hybrid – the group consisting of (male, hybrid) and (female, hybrid).
- When we combine one or more individual outcomes in a collection, we are creating what is known as **event.**

An **event** is any collection of outcomes from the sample space of a chance experiment.

Complement, Union & Intersection

Let **A** and **B** denote two events.

- The event **not A** (or **complement of A**) consists of all experimental outcome that are not in event **A**, denoted by

$$\boldsymbol{A', A^c, \overline{A},}$$

- The event **A or B** (or **the union of two events**) consists of all experimental outcome that are in at least one of the two events, that is, in A or in B or in both of these, denoted by

$$\boldsymbol{A \cup B}$$

- The event **A and B** (or **the intersection of two events**) consists of all experimental outcome that are in both A and B, denoted by

$$\boldsymbol{A \cap B}$$

Example - Complement

Suppose a six-sided die is rolled. The event that the die would land on an even number would be

$$E = \{2, 4, 6\}$$

What would the event be that is the die **NOT** landing on an even number?

$$E^C = \{1, 3, 5\}$$

The **sum** of the probabilities of complementary events equals **ONE**.

Example - Union

Suppose a six-sided die is rolled. The event that the die would land on an even number would be

$$E = \{2, 4, 6\}$$

The event that the die would land on a prime number would be

$$P = \{2, 3, 5\}$$

What would be the event **E or P** happening?

$$E \text{ or } P = \{2, 3, 4, 5, 6\}$$

Example - Intersection

Suppose a six-sided die is rolled. The event that the die would land on an even number would be

E = {2, 4, 6}

The event that the die would land on a prime number would be

P = {2, 3, 5}

What would be the event **E and P** happening?

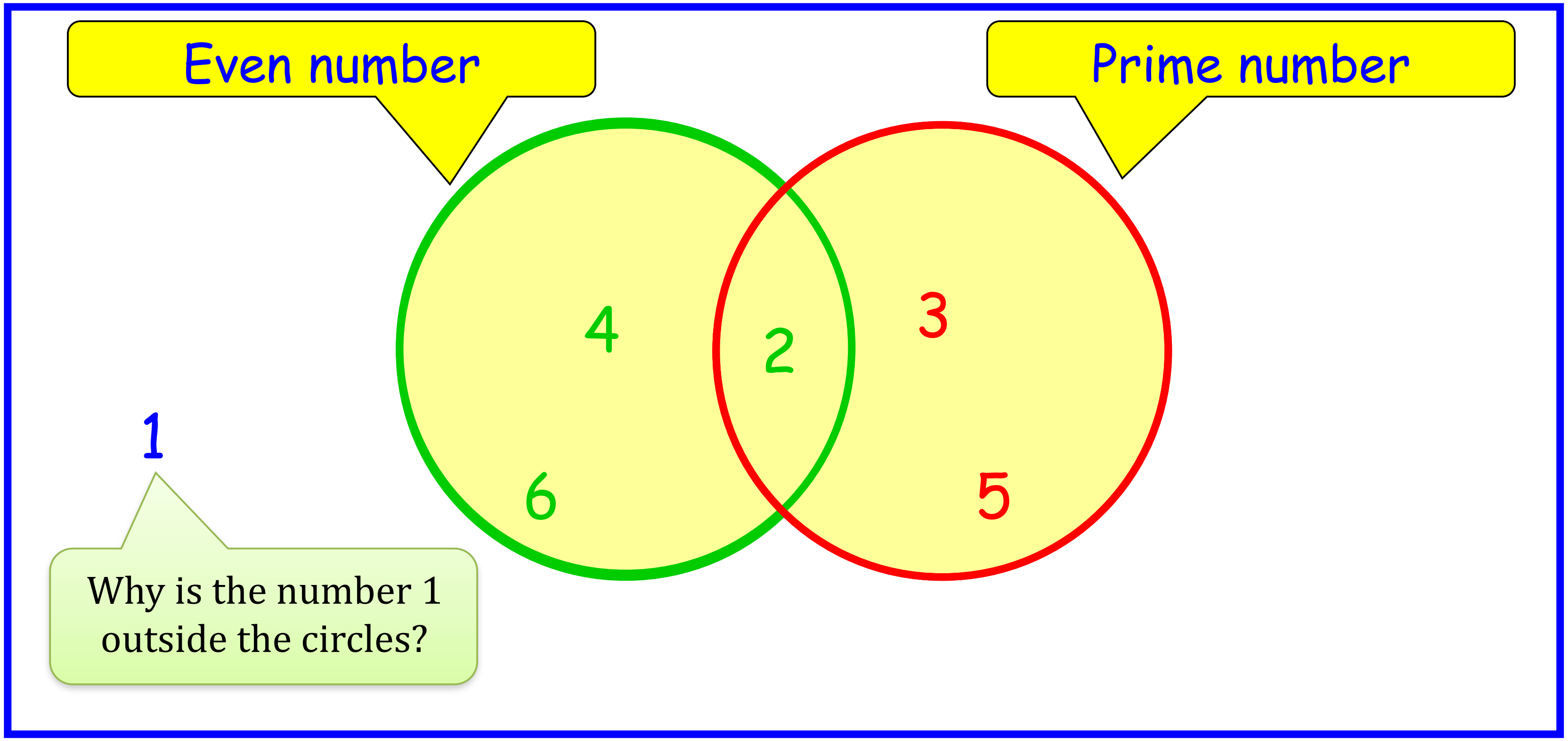
E and P = {2}

Venn Diagram

Let's revisit rolling a die and getting an even or a prime number . . .

E or P = {2, 3, 4, 5, 6}

Another way to represent this is with a Venn Diagram.



Approach to Probability

When the outcomes in a sample space are equally likely, the probability of an event **E**, denoted by ***P*(*E*)**, is the **ratio** of the number of outcomes favorable to **E** to the total number of outcomes in the sample space.

$$P(E) = \frac{\text{no. of outcome favorable to E}}{\text{no. of outcome in the sample space}}$$

Examples: flipping a coin, rolling a die, etc.

Probability - Rules

- Fundamental Properties of Probability

Property 1. Legitimate Values

For any event *E*, $0 \leq p(E) \leq 1$0

Property 2. Sample space

If S is the sample space, P(S) = 1

Property 3: Addition

If two events *E* and *F* are disjoint,

$P(E \text{ or } F) = P(E) + P(F)$

Property 4: Complement

For any event *E*, $P(E) + P(\text{not } E) = 1$

Exercise #1

The student council for a school of science and math has one representative from each of the five academic departments: biology (B), chemistry (C), mathematics (M), physics (P), and statistics (S). Two of these students are to be randomly selected for inclusion on a university-wide student committee (by placing five slips of paper in a bowl, mixing and drawing out two of them).

a) What are the 10 possible outcomes?
b) From the description of the selection process, all outcomes are equally likely. What is the probability of each event?
c) What is the probability that one of the committee members is the statistics department representative?
d) What is the probability that both committee members come from laboratory science departments?

Exercise #2

A large department store offers online ordering. When a purchase is made online, the customer can select one of four different delivery options: expedited overnight delivery, expedited second-business-day delivery, standard delivery, or delivery to the nearest store for customer pick-up. Consider the chance experiment that consists of observing the selected delivery option for a randomly selected online purchase.

a) What are the events that make-up the sample space for this experiment?

b) Suppose that the probability of an overnight delivery selection is 0.1, the probability of a second-day delivery selection is 0.3, and the probability of a standard-delivery selection is 0.4. Find the following probabilities:

i) The probability that a randomly selected online purchase selects delivery to the nearest store for customer pick-up.

ii) The probability that the customer selects a form of expedited delivery.

iii) The probability that either standard delivery or delivery to the nearest store is selected.

CHAPTER 4
(Part 2)

Random Variables & Probability Distribution

Random Variables

- A numerical variable whose value depends on the outcome of a chance experiment.
- It associates a numerical value with each outcome of a chance experiment.
- Two types of random variables:
  1) Discrete
  2) Continuous

Discrete Random Variables

- Discrete – its set of possible values is a collection of isolated points along a number line.

This is typically a "count" of something

- Continuous - its set of possible values includes an entire interval on a number line.

This is typically a "measure" of something

Example

Suppose that a counsellor plans to select a random sample of 50 seniors at a large high school and to ask each student in the sample whether he or she plans to attend college after graduation. The process of sampling is a chance experiment. The sample space for this experiment consists of all different possible random samples of size 50 that might result (there is a very large number of these) and for simple random sampling, each of these outcomes is equally likely.

Let x is the random variable.

- x represent the number of successes in the sample (who plans to attend college after graduation).
- Thus, x is discrete random variable (counting).

Example

A point is randomly selected on the surface of a lake that has a maximum depth of 100 feet. Let y be the depth of the lake at the random chosen point.

- The possible value of y : measurement in feed of the depth of the lake.
- Thus, y is continuous random variable.

Example

Consider an experiment in which the type of book, print (P) and digital (D), chosen by each of three successive customers making a purchase from on an online bookstore is recorded. Define a **random variable** x by

$$x = \text{ number of customers purchasing a book in digital format}$$

The experimental outcome can be abbreviated, e.g., DPD (first and third customer purchase a digital book).

Outcome	PPP	DPP	PDP	PPD	DDP	DPD	PDD	DDD
x value	0	1	1	1	2	2	2	3

There are only 4 possible x values: 0, 1, 2 and 3. Thus, x is discrete random variable.

Probability Distribution For DISCRETE Random Variables

Definition

- Probability distribution is a model that describes the long-run behaviour of a variable.
- The mathematical definition of a discrete probability distribution is : $p(x)$
- If x =2, we write $p(2)$ in place of $p(x=2)$
- $p(5)$ denotes the probability that $x=5$.

- It give the possibility associated with each possible x value.
- Each probability is the long-run relative frequency of occurrence of the corresponding x value when the chance experiment is performed a very large number of times.
- Common way to display a probability distribution for discrete random variable: table, histogram, formula.

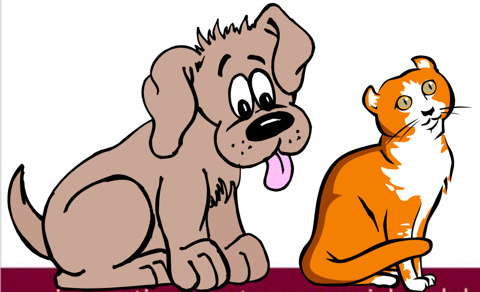
Example

In a Wolf City, regulations prohibit no more than five dogs or cats per household.

Let, x = the number of dogs and cats in a randomly selected household in Wolf City.

Then, the Department of Animal Control has collected data over the course of several years.

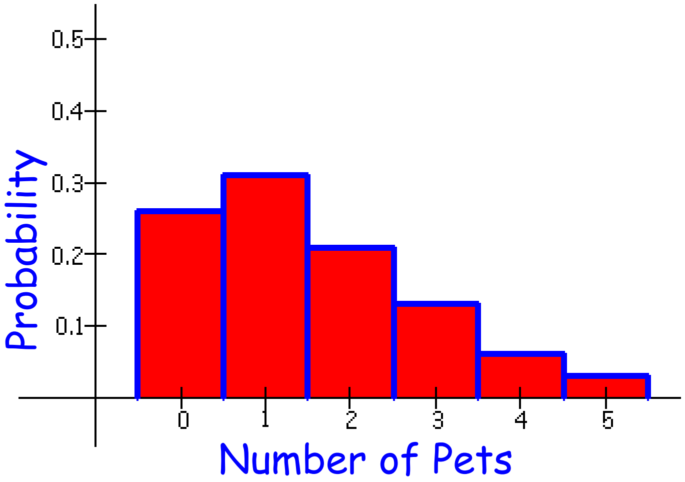
They want to estimate the long-run probabilities for the values of x.



- The results as follows:

x	0	1	2	3	4	5
$p(x)$	0.26	0.31	0.21	0.13	0.06	0.03

This is called a discrete probability distribution. It can also be displayed in a histogram with the probability on the vertical axis.



Properties of Discrete Probability Distributions

1) For every possible x value,

$$0 \leq p(x) \leq 1.$$

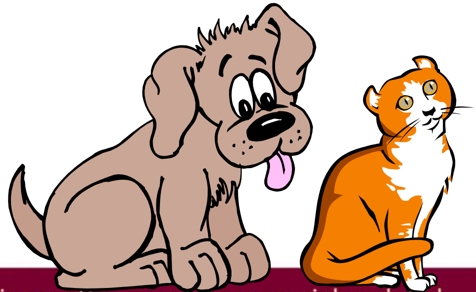
2) $$\sum_{\text{all } x \text{ values}} p(x) = 1$$

Example

Refer to previous example (Dogs and Cats) .

x	0	1	2	3	4	5
$p(x)$	0.26	0.31	0.21	0.13	0.06	0.03

What is the probability that a randomly selected household in Wolf City has at most 2 pets?



$$p(x \leq 2) = p(0) + p(1) + p(2)$$

$$= 0.26 + 0.31 + 0.21 = 0.78$$

x	0	1	2	3	4	5
$p(x)$	0.26	0.31	0.21	0.13	0.06	0.03

What is the probability that a randomly selected household in Wolf City has less than 2 pets?

$$p(x < 2) = p(0) + p(1)$$

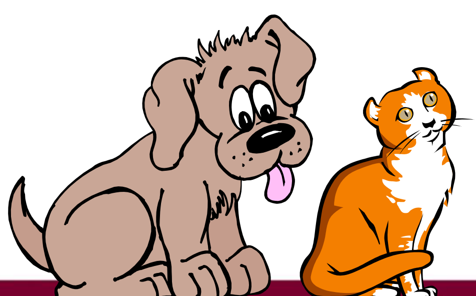
$$= 0.26 + 0.31 = 0.57$$

x	0	1	2	3	4	5
$p(x)$	0.26	0.31	0.21	0.13	0.06	0.03

What is the probability that a randomly selected household in Wolf City has <u>more than 1 but no more than 4 pets</u>?

$$p(1 < x \leq 4) = p(1) + p(2) + p(3)$$

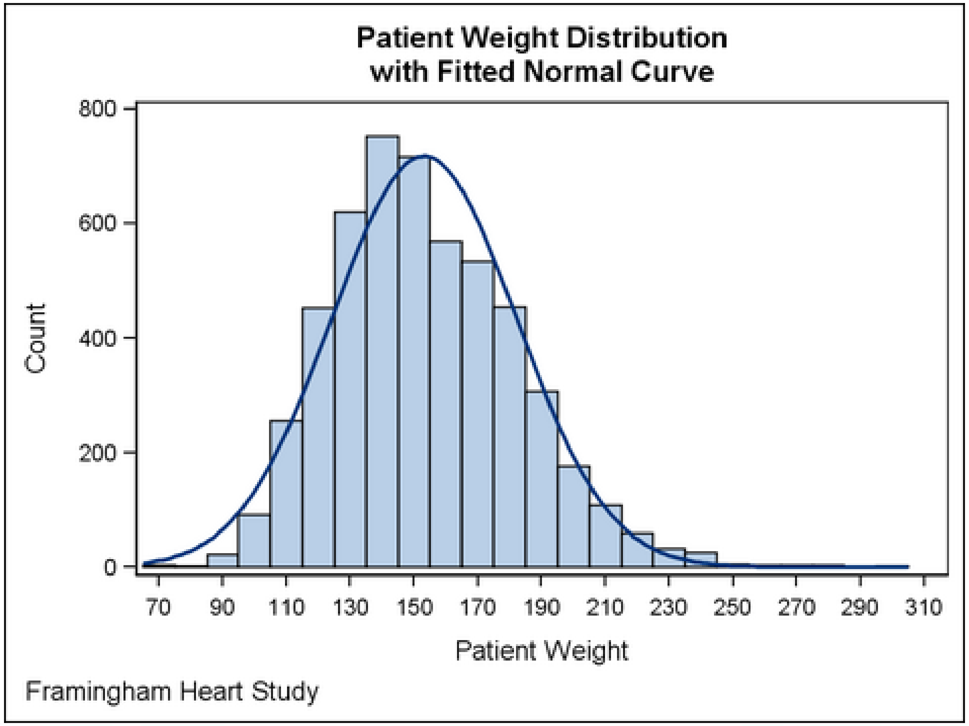
$$= 0.21 + 0.13 + 0.06 = 0.40$$



Probability Distribution For CONTINUOUS Variables

Definition:
Continuous Random Variable

- A continuous random variables is a random variable where the data can take infinitely many values.
- A continuous random variable "x" takes all values in a given interval of numbers.

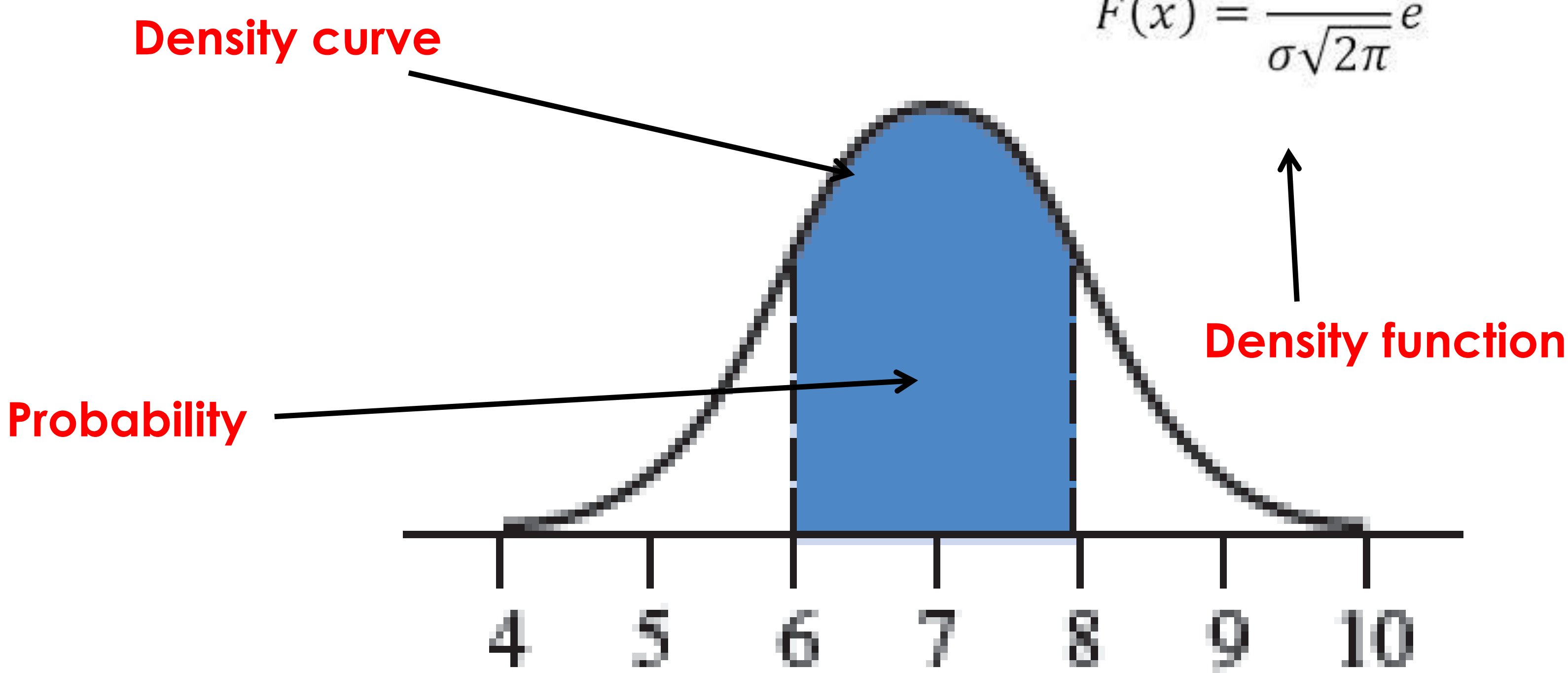


Definition :
Probability Distribution for Continuous Random Variable

- It is specified by a curve called a density curve.
- The function that describes this curve is denoted by $f(x)$ and is called the density function.
- The probability of observing a value in a particular interval is the area under the curve and above the given interval.

Normal Probability Density Function

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2 / 2\sigma^2}$$



Properties of Continuous probability distributions

1. $f(x) \geq 0$ (the curve cannot dip below the horizontal axis).

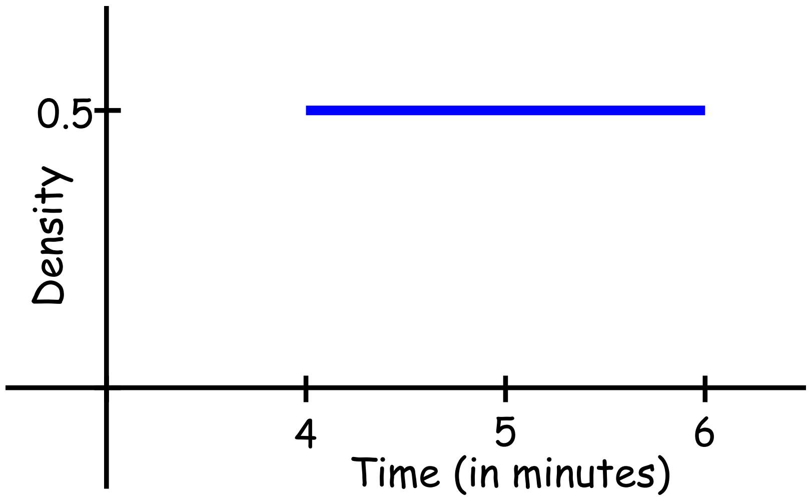
2. The total area under the density curve equals one.

Example

Suppose x is a continuous random variable defined as the amount of time (in minutes) taken by a clerk to process a certain type of application form. Suppose x has a probability distribution with density function:

$$f(x) = \begin{cases} .5 & 4 < x < 6 \\ 0 & \text{otherwise} \end{cases}$$

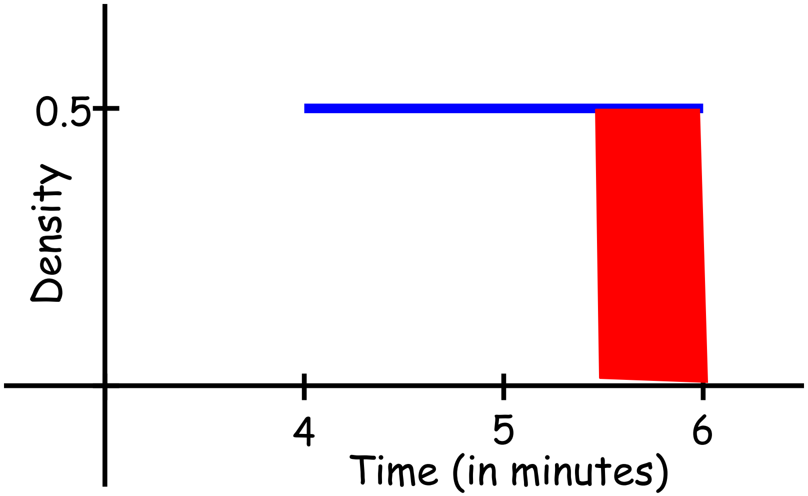
The following is the graph of $f(x)$, the density curve:



Example (cont.)

What is the probability that it takes more than 5.5 minutes to process the application form?

$$p(x > 5.5) = 0.5(0.5) = 0.25$$



Other Density Curves

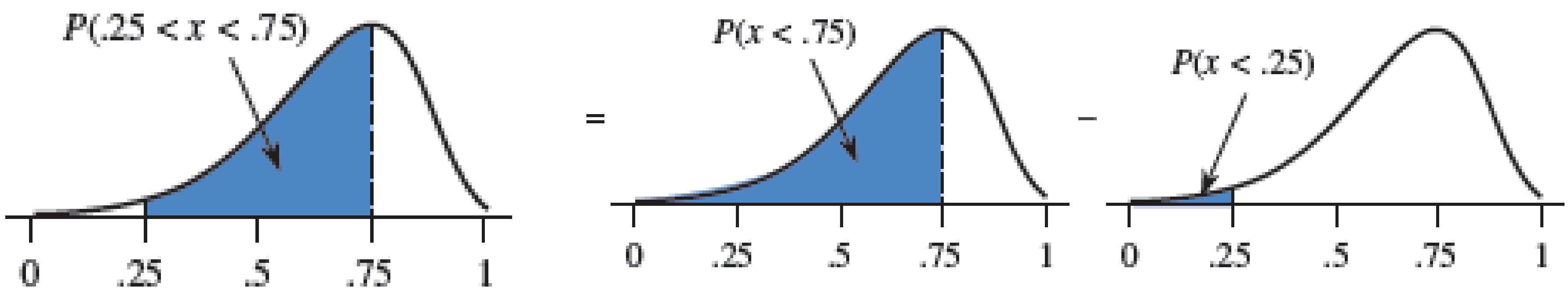
Some density curves resemble the one below. Integral calculus is used to find the area under the these curves.

We will use tables (with the values already calculated).

The probability that a continuous random variable x lies between a lower limit a and an upper limit b is

$p(a < x < b)$ = (cumulative area to the left of b) – (cumulative area to the left of a)

$$p(a < x < b) = p(x < b) - p(x < a)$$



- Other continuous distribution

We will use tables for certain distribution (with the values already calculated).

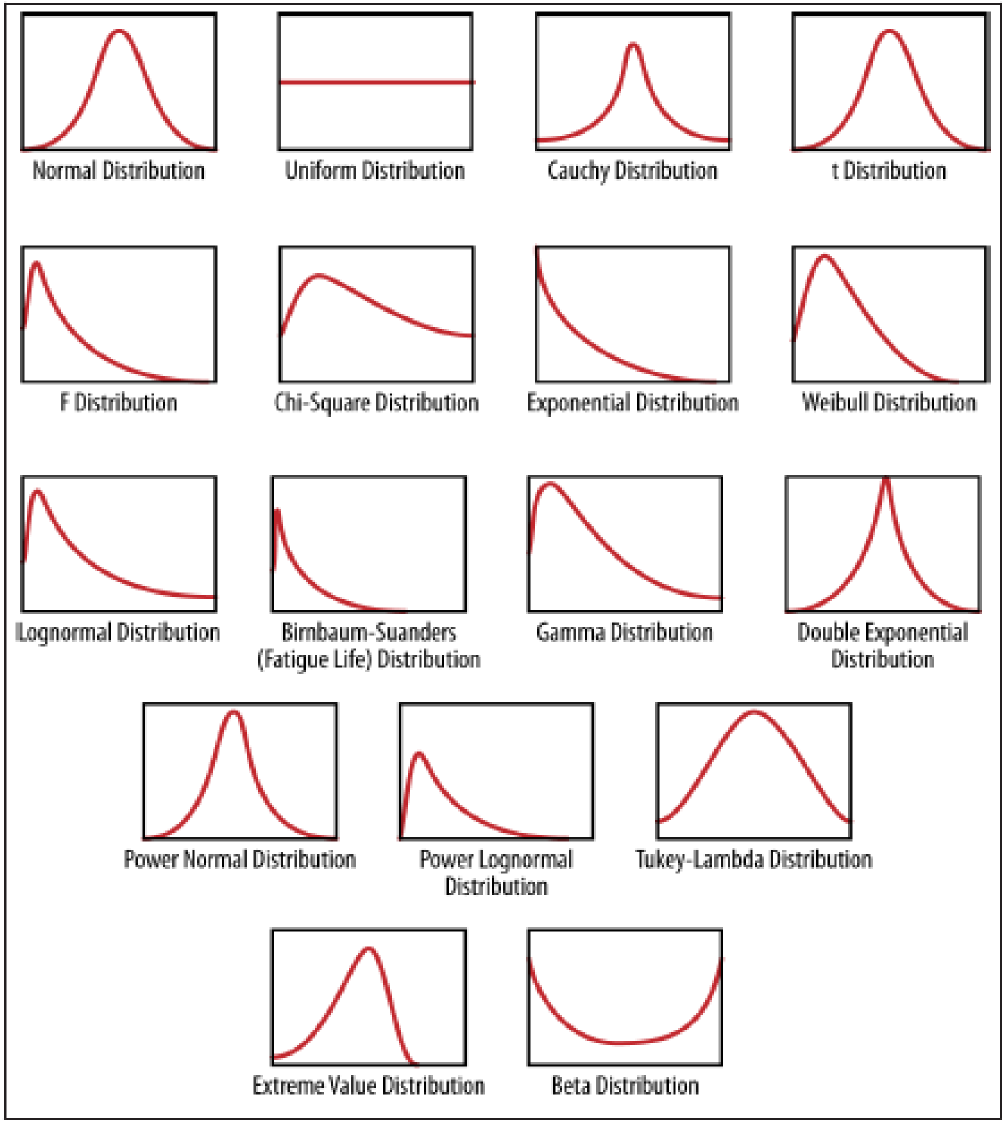


Figure 2-1. A bunch of continuous density functions (aka probability distributions)

CHAPTER 4
(Part 3)

Discrete & Continuous Probability Distribution

Special Distributions

- Discrete Distribution:
  - Binomial
  - Negative Binomial
  - Geometric
- Continuous Distribution:
  - Normal

Discrete Distribution

Binomial Distribution

- An experiments often consists of repeated trials, each with two possible outcomes that may be labeled **success** or **failure**.
- **Example**: The testing items as they come off an assembly line, where each trial may indicate a defective or a non-defective item.
- The process is referred to as a **Bernoulli process**.
- Each trial is called a **Bernoulli trial**.

Binomial Distribution (cont'd)

Properties of a Bernoulli process:

1. There are a fixed number of trials.
2. Each trial results in one of only two possible outcomes, labeled success (S) or failure (F).
3. Outcomes of different trials are independent.
4. The probability that a trial results in success is the same for all trials.

Binomial Distribution (cont'd)

- The number X of success in n Bernoulli trials is called a binomial random variable.
- The probability distribution of this discrete random variable is called the **binomial distribution**.
- **Formula**:

A Bernoulli trial can result in a success with probability p and a failure with probability q = 1-p. Then the probability distribution of the binomial random variable X, the number of successes in n independent trials, is

$$b(x;n,p) = \binom{n}{x} p^x q^{n-x}, x = 0, 1, 2, \dots, n$$

- The **mean** and **variance** of the binomial distribution:

$$\mu = np \; ; \; \sigma^2 = npq$$

Example

The probability that a certain kind of component will survive a shock test is ¾. Find the probability that exactly 2 of the next 4 components tested survive.

Solution:

Assuming that the tests are independent and p = ¾ for each of the 4 tests, we obtain

$$b(2;4,{}^{3}/_{4}) = \binom{4}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2 = \frac{27}{128}$$

Example

A coin is tossed four times. Find the mean and variance of the number of heads that will be obtained? Given the distribution as follows.

No of head, x	0	1	2	3	4
Probability	1/16	4/16	6/16	4/16	1/16

Solution:

$$n = 4, p = \frac{1}{2}; q = \frac{1}{2}$$

$$.: \mu = 4\left(\frac{1}{2}\right) = 2;$$

$$.: \sigma^2 = 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 1$$

Negative Binomial Distribution

If repeated independent trials can result in a success with probability *p* and a failure with probability $q = 1 - p$, then the probability distribution of the random variable *x*, the number of the trial on which the *k*-th success occurs, is

$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}, x = k, k+1, k+2, \ldots.$$

Example

A scientist inoculates mice, one at a time, with a diseases germ until he finds 2 that have contracted the diseases. If the probability of contracting the diseases is 1/6, what is the probability that 8 mice are required?

Solution:

$$b^*(8;2,0.1667) = \binom{8-1}{2-1}(0.1667)^2(1-0.1667)^{8-2} = 0.0651$$

Geometric Distribution

Geometric distribution is in a sense **INFINITE.** They're asking you what is the probability **UNTIL** the first success happens.

Example:

Malik throws a dice until he gets a '4'.

What is the probability of him getting the first '4' in the 6^{th} throw? How to calculate the probability?

Geometric Distribution (cont'd)

Geometric Probability Distribution function:

If x is a random variable with probability of success = p for each trial, then

$$g(x; p) = (1-p)^{x-1}p, x = 1, 2, 3, \ldots.$$

Geometric Distribution (cont'd)

Formulas for mean and standard deviation of a geometric distribution:

$$\text{mean: } \mu_x = \frac{1}{p}$$

$$\text{std. dev: } \sigma_x = \sqrt{\frac{(1-p)}{p^2}}$$

Example

A product produced by a machine has a 3% defective rate. What is the probability that the first defective occurs in the fifth(5^{th}) item inspected?

Solution:

$$p(x=5)=(1-0.03)^{5-1}(0.03)$$
$$=0.027$$

Exercise#1

It is claimed that 15% of the ducks in a particular region have patent schistosome infection. Suppose that seven ducks are selected at random. Let x equals the number of ducks that are infected.

a) Assuming independence trials, how is x distributed?

b) Find the probability that:

i) At most two ducks are infected.

ii) One duck is not infected.

c) A new drug is proposed to treat the schistosome infection. Therefore, the infected duck is randomly selected. What is the probability that the first duck infected is found at the fifth trial?

Exercise #2

In an NBA (National Basketball Association) championship series, the team that wins four games out of seven is the winner. Suppose that teams A and B face each other in the championship games and that team A has probability 0.55 of winning a game over team B.

a) What is the probability that team A will win the series in 6 games?

b) What is the probability that team A will win the series?

c) If teams A and B were facing each other in a regional playoff series, which is decided by winning three out of five games, what is the probability that team A would win the series?

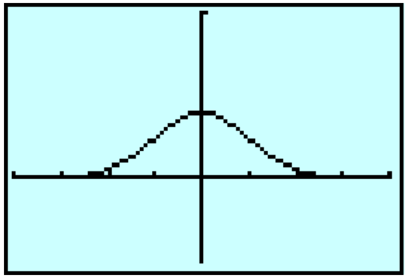
Exercise #3

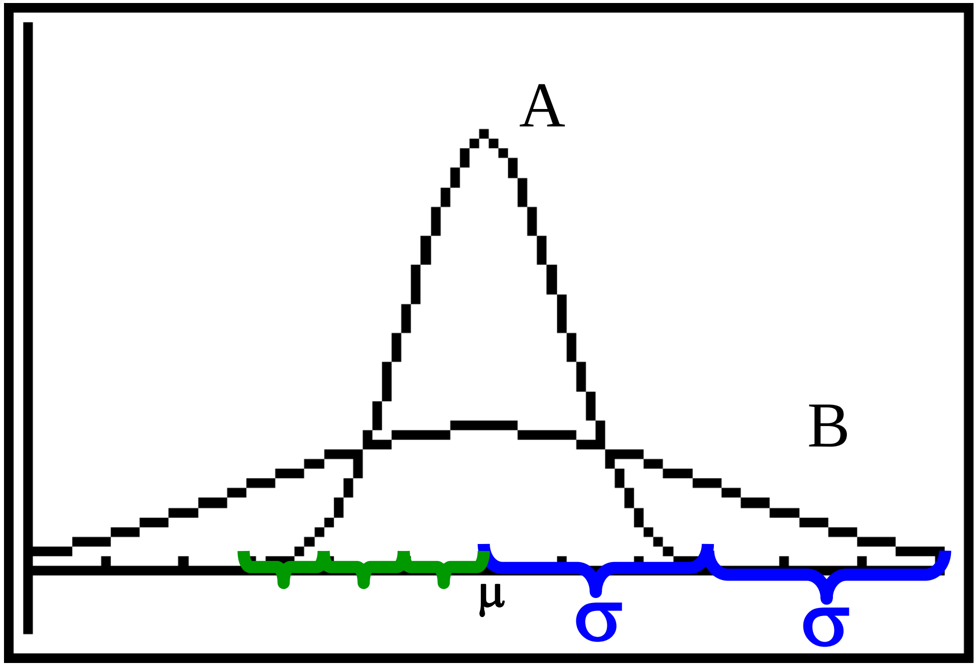
A company fabricates special-purpose robots and records show that the probability is 0.10 that the one of its new robots will require repairs during confirmation tests. What is the probability that the eighth robot it builds in a month is the first one to require repairs?

Continuous Distribution

Continous Distribution

- Symmetrical bell-shaped (unimodal) density curve defined by μ and σ
- Area under the curve equals 1.
- Probability of observing a value in a particular interval is calculated by finding the area under the curve.
- As σ increases, the curve flattens & spreads out.
- As σ decreases, the curve gets taller and thinner.



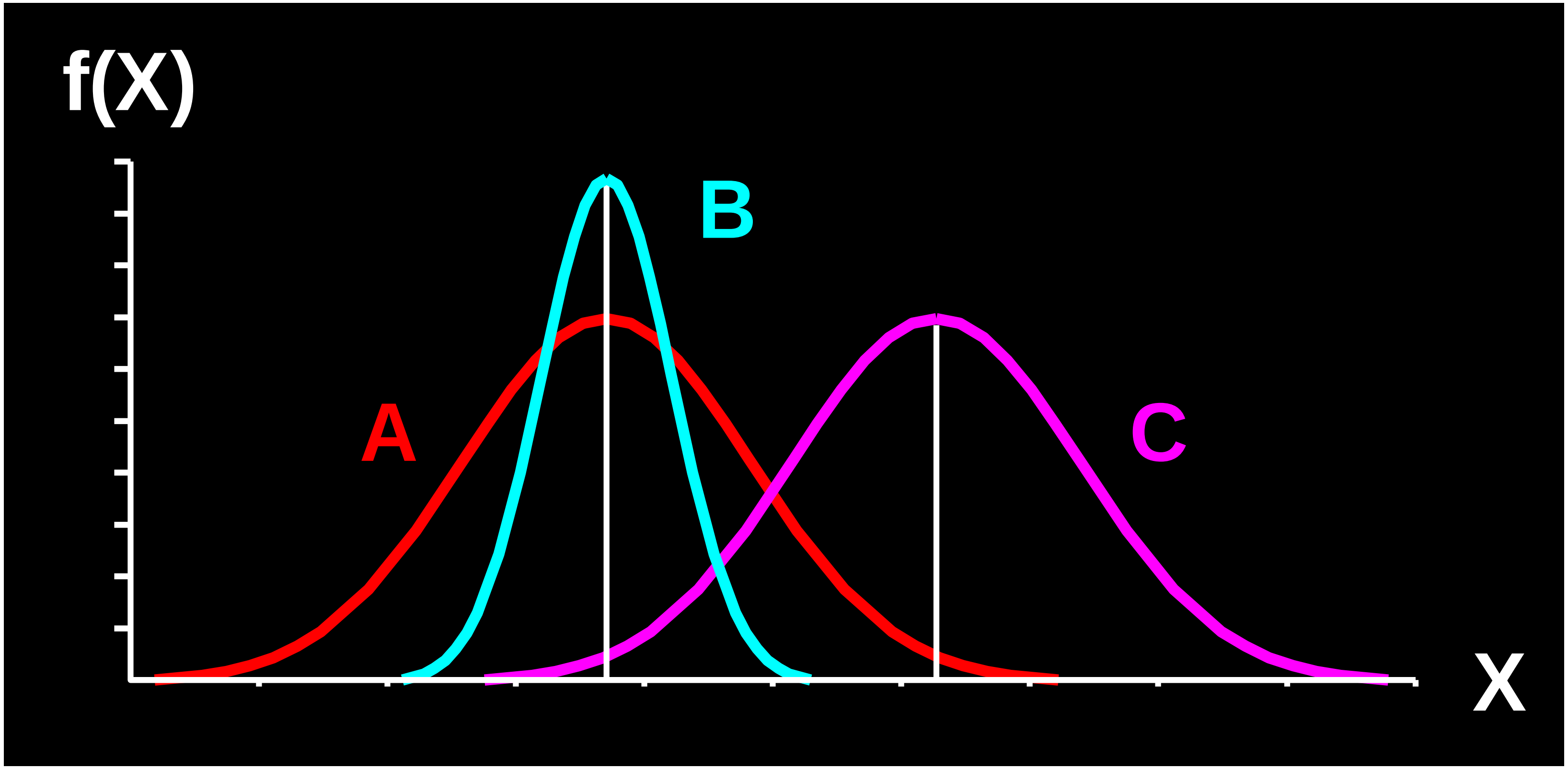


Do these two normal curves have the same mean?

Which normal curve has a standard deviation of 3?

Which normal curve has a standard deviation of 1?

Effect of Varying Parameters (μ & σ)



Notation

$$x \text{ is } N(\mu,\sigma)$$

The random variable x has a normal distribution (N) with mean, μ and standard deviation, σ.

Example:

$$x \text{ is } N(40,1)$$

$$x \text{ is } N(10,5)$$

$$x \text{ is } N(50,3)$$

Probability Density Function

Normal Distribution

Parameters: μ and σ

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}$$

for $-\infty < x < \infty$

x = value of random variable ($-\infty < x < \infty$)

σ = Population Standard Deviation

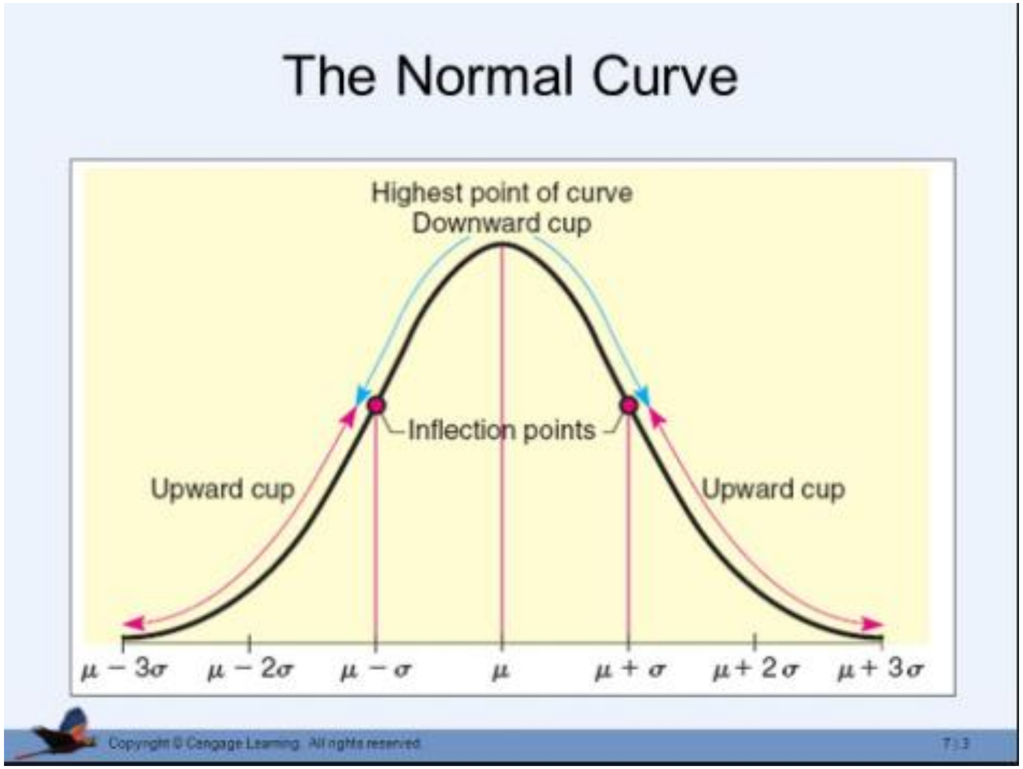
π = 3.14159

e = 2.71828

μ = Mean of Random Variable x

Normal Curve

Notice that the normal curve is curving downwards from the center (mean) to points that are one standard deviation on either side of the mean. At those points, the normal curve begins to turn upward.



Standard Normal Distribution

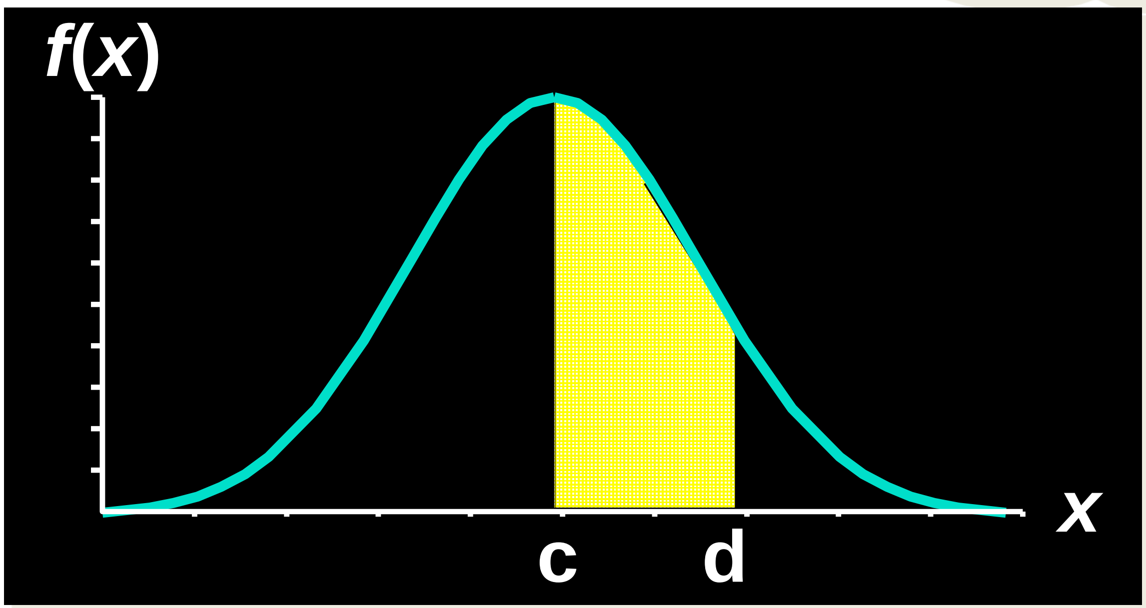
- Is a normal distribution with $\mu = 0$ and $\sigma = 1$

- It is customary to use the letter z to represent a variable whose distribution is described by the **standard normal curve** (or z curve).

Normal Distribution Probability

Probability is area under curve!

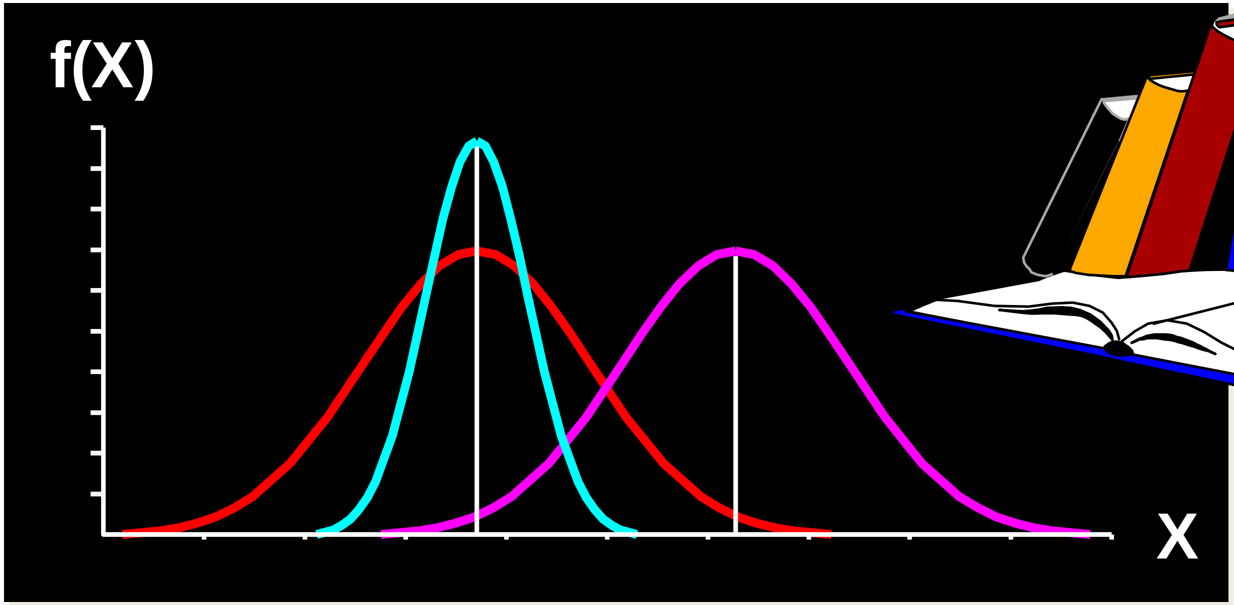
$$P(c \le x \le d) = \int_c^d f(x)\,dx\ ?$$



Infinite Number of Distribution Tables

Normal distributions differ by mean & standard deviation.

Each distribution would require its own table.

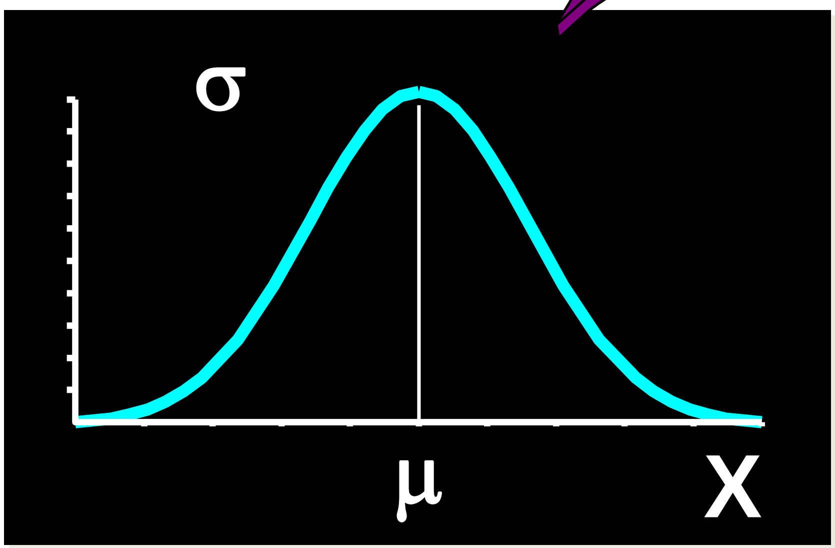


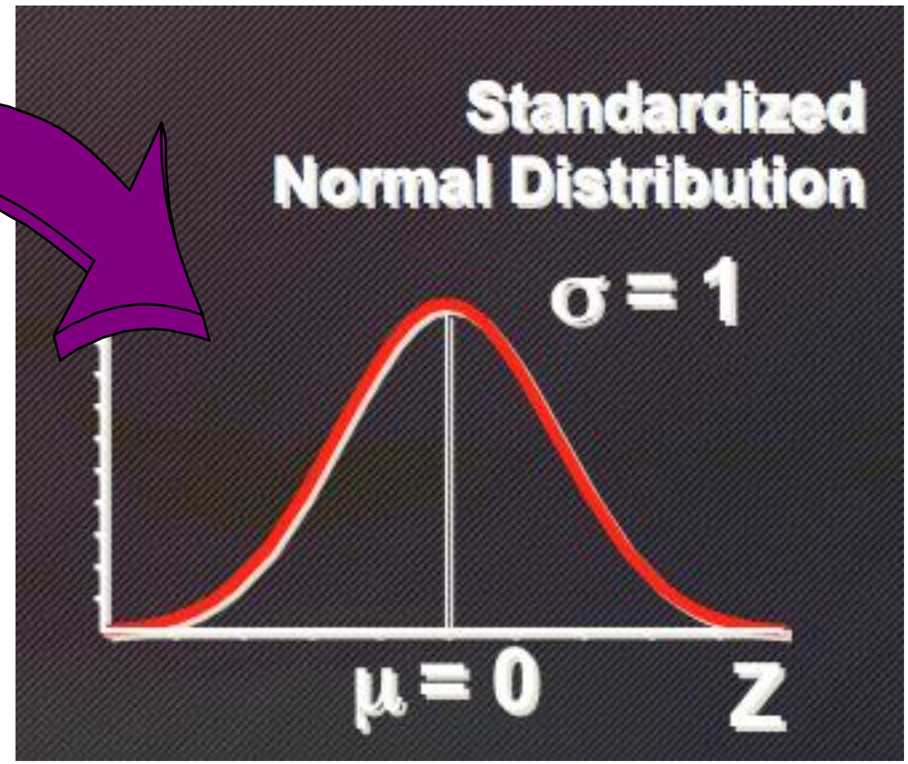
That's an *infinite* number!

Standardize the Normal Distribution

$$Z = \frac{X - \mu}{\sigma}$$

Normal Distribution



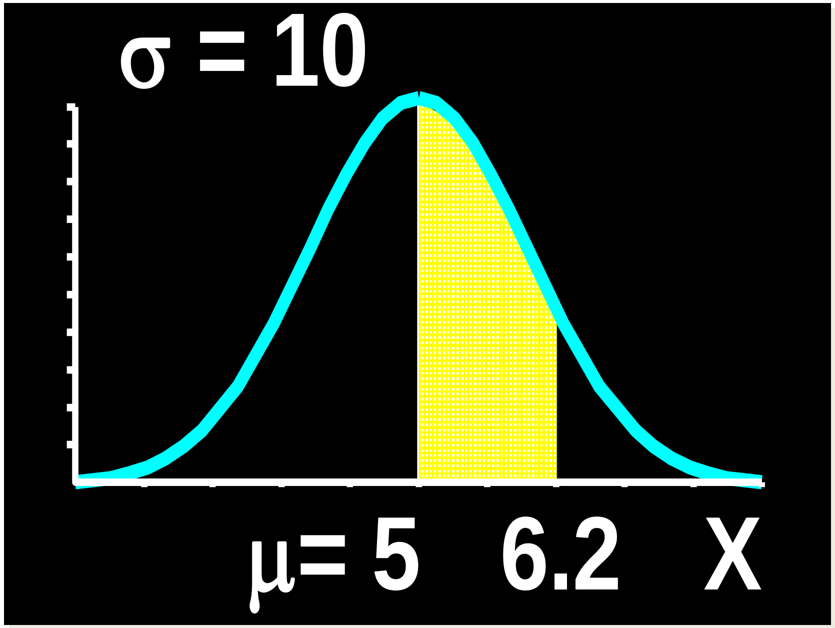


One table!

Example #1

What is the probability of P(X=6.2), given that the mean is 5 and std. deviation is 10?.

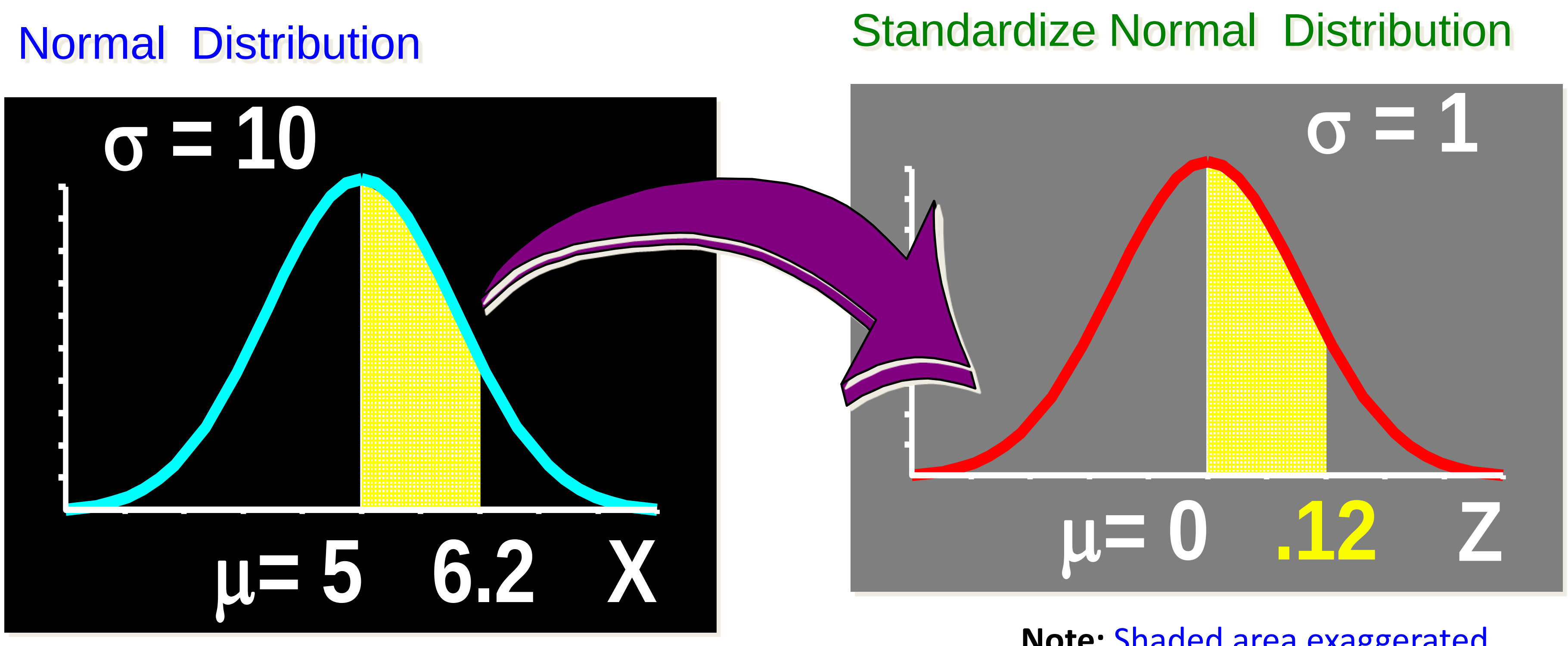
Normal Distribution



Example #1 -Solution

Step 1: Find the Z value.

$$Z = \frac{X - \mu}{\sigma} = \frac{6.2 - 5}{10} = 0.12$$

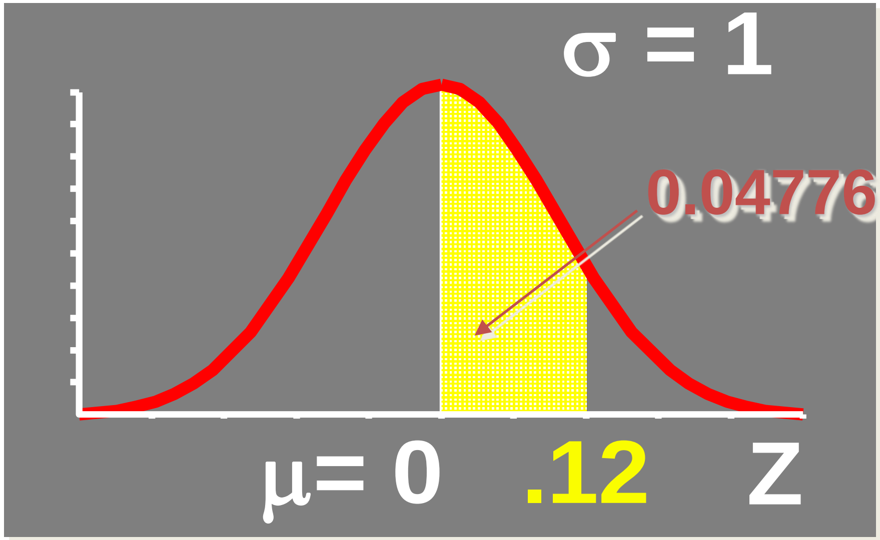


Note: Shaded area exaggerated.

Step 2: Refer to standard Normal Dist. Table to locate the Z value.

Standardized Normal Probability Table (Portion)

Z	.00	.01	.02
0.0	.50000	.50399	.50798
0.1	.53983	.54380	.54776
0.2	.57926	.58317	.58706
0.3	.61791	.62172	.62552
0.4	.65542	.65910	.66276



P(Z=0.12) = 0.54776 - 0.5 = 0.04776

Example #2: Find P(3.8 ≤ *X* ≤ 5)

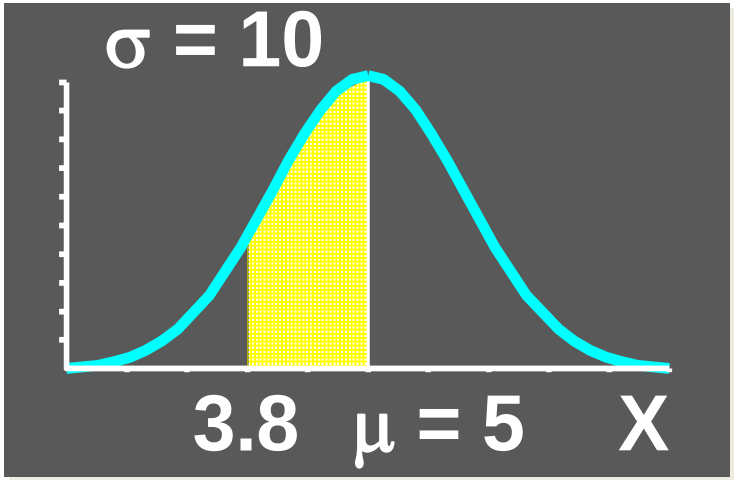
$$Z = \frac{X-\mu}{\sigma} = \frac{3.8-5}{10} = -.12$$

Z	.00	.01	.02
....			
-0.4	.34458	.34090	.33724
-0.3	.38209	.37828	.37448
-0.2	.42074	.41683	.41294
-0.1	.46017	.45620	.45224
-0.0	.50000	.49601	.49202

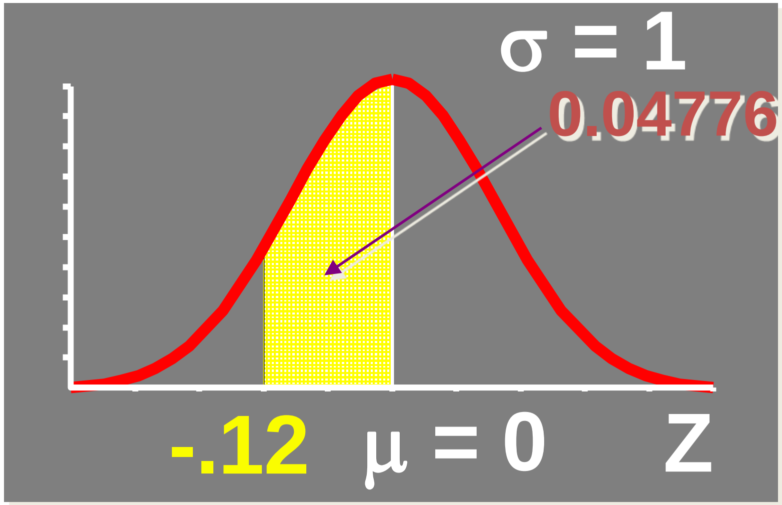
P(Z=0.12) = 0.5 – 0.45224

= 0.04776

Normal Distribution

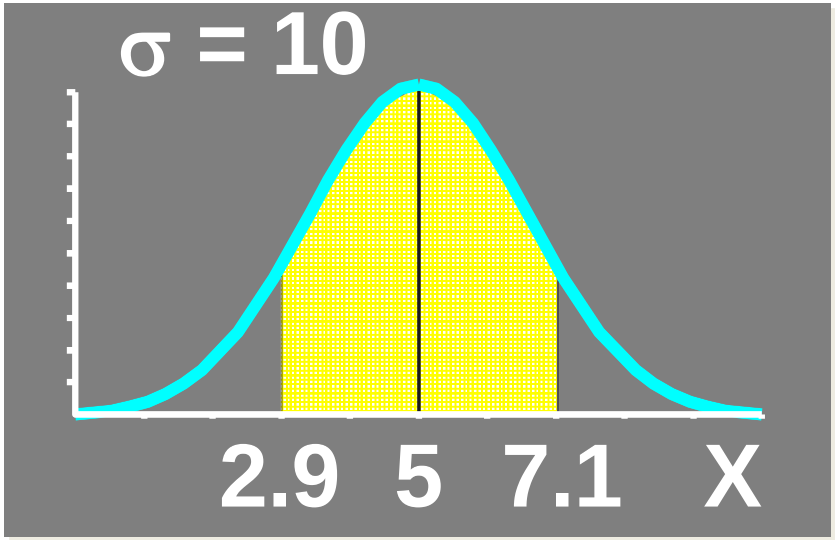


Standardize Normal Distribution



Example #3: Find $P(2.9 \le X \le 7.1)$

Normal Distribution



$$Z = \frac{X - \mu}{\sigma} = \frac{2.9 - 5}{10} = -.21$$

$$Z = \frac{X - \mu}{\sigma} = \frac{7.1 - 5}{10} = .21$$

$$Z = \frac{X-\mu}{\sigma} = \frac{2.9-5}{10} = -0.21$$

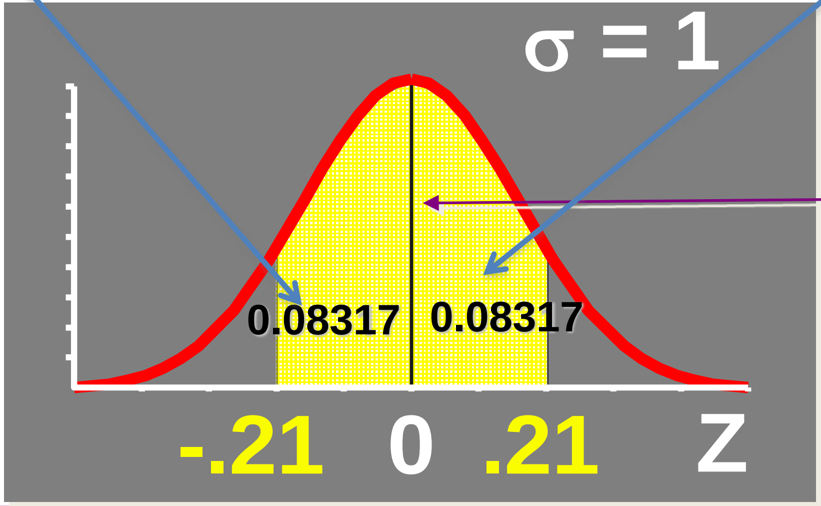
Z	.00	.01	.02
....			
-0.4	.34458	.34090	.33724
-0.3	.38209	.37828	.37448
-0.2	.42074	.41683	.41294
-0.1	.46017	.45620	.45224
-0.0	.50000	.49601	.49202

P(Z=-0.21) = 0.5 – 0.41683= 0.08317

$$Z = \frac{X-\mu}{\sigma} = \frac{7.1-5}{10} = 0.21$$

Z	.00	.01	.02
0.0	.50000	.50399	.50798
0.1	.53983	.54380	.54776
0.2	.57926	.58317	.58706
0.3	.61791	.62172	.62552
0.4	.65542	.65910	.66276

P(Z=0.21) = 0.58317 – 0.5= 0.08317



0.08317+0.08317

=0.16634

Example #4: Find $P(X \geq 8)$

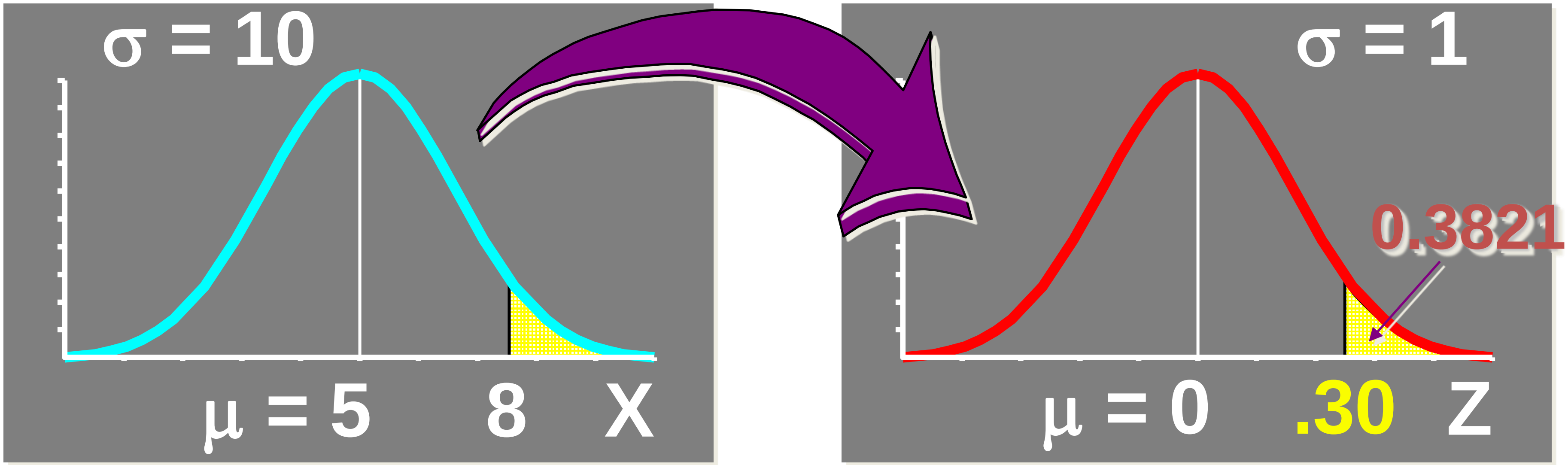
$$Z = \frac{X-\mu}{\sigma} = \frac{8-5}{10} = .30$$

Z	.00	.01	.02
0.0	.50000	.50399	.50798
0.1	.53983	.54380	.54776
0.2	.57926	.58317	.58706
0.3	.61791	.62172	.62552
0.4	.65542	.65910	.66276

P(Z=0.30) = 1 - 0.61791 = 0.3821

Normal Distribution

Standardize Normal Distribution



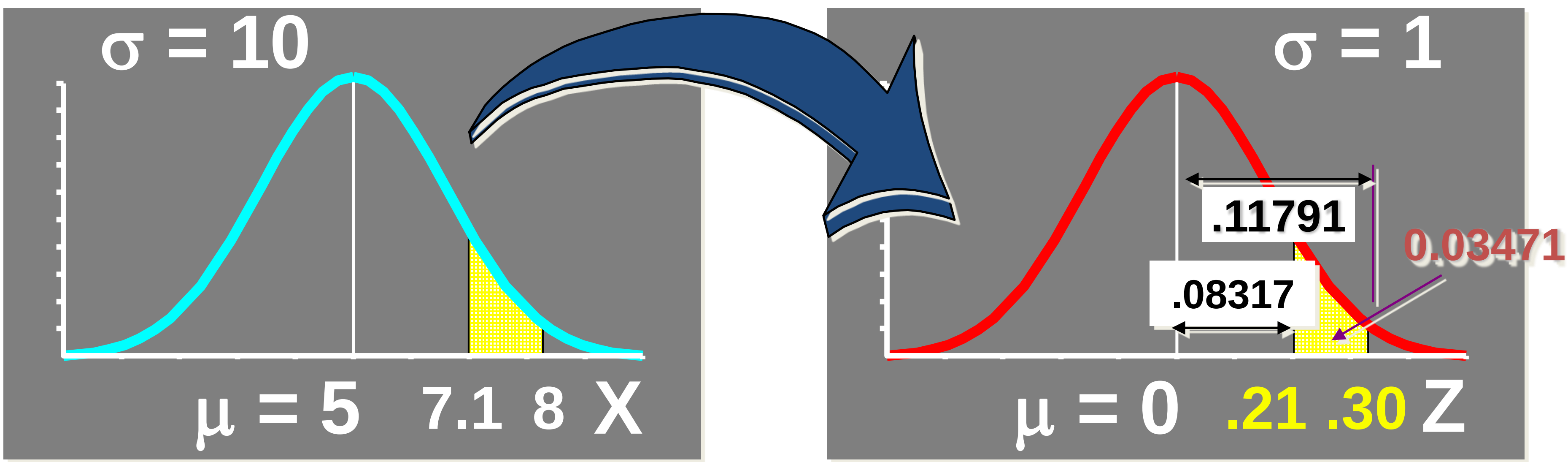
Example #5: Find P(7.1 ≤ *X* ≤ 8)

$$Z = \frac{X-\mu}{\sigma} = \frac{7.1-5}{10} = 0.21$$ → P(Z=0.21) = 0.58317 – 0.5= 0.08317

$$Z = \frac{X-\mu}{\sigma} = \frac{8-5}{10} = 0.30$$ → P(Z=0.30) = 0.61791- 0.5 = 0.11791

P(0.21 ≤ *Z* ≤ 0.30) = 0.11791 – 0.0832 = 0.03471

Normal Distribution | Standardize Normal Distribution

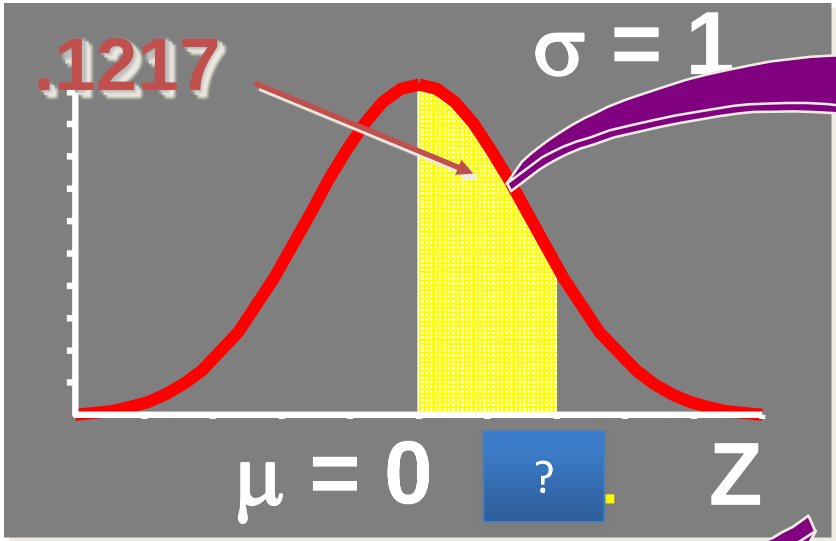


Finding Z Values for Known Probabilities

If Z greater than mean, what is Z given that P(Z) = 0.12172?

P(Z=?) = a - 0.5 = 0.12172 => 0.12172 + 0.5 = 0.62172

Standardize Normal Distribution



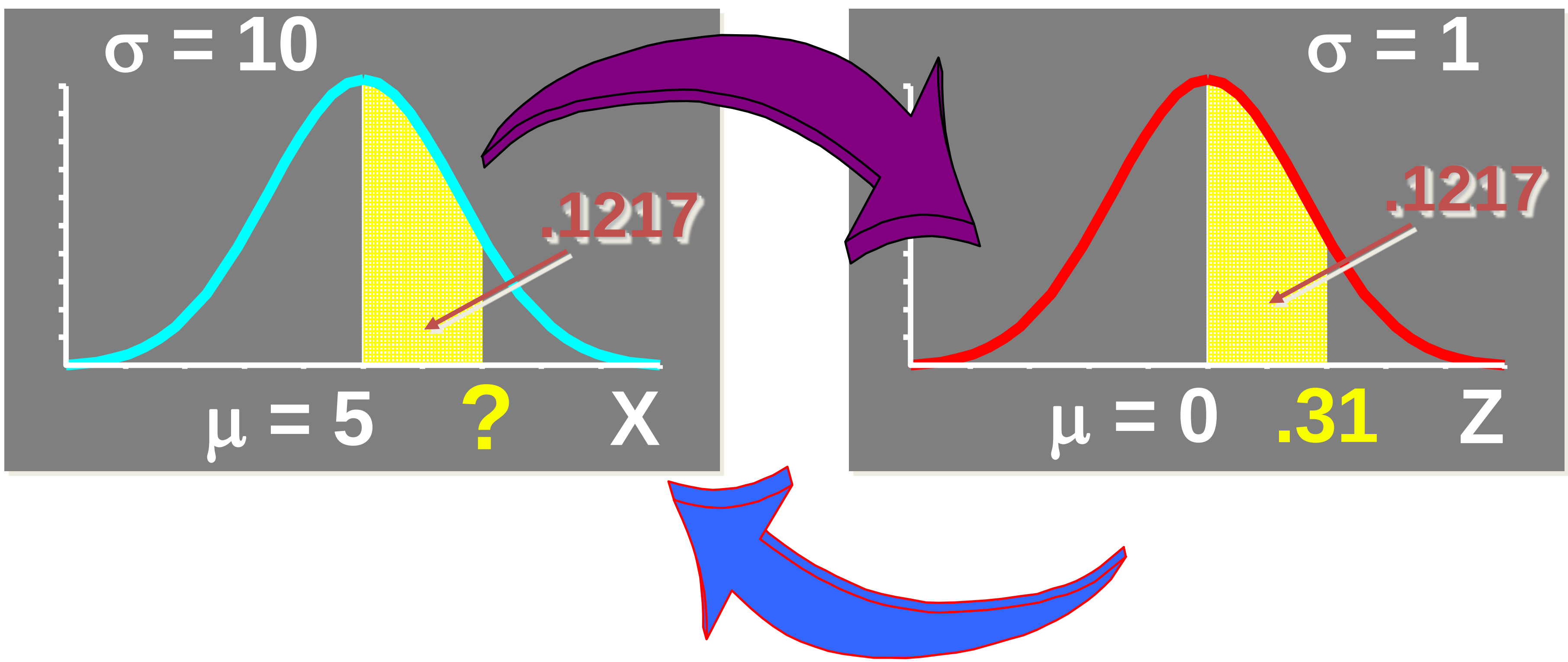
Standardized Normal Probability Table (Portion)

Z	.00	.01	.02
0.0	.50000	.50399	.50798
0.1	.53983	.54380	.54776
0.2	.57926	.58317	.58706
0.3	.61791	.62172	.62552
0.4	.65542	.65910	.66276

Note:
If Z less than mean, than Z value has negative sign.

Normal Distribution

Standardize Normal Distribution



$$X = \mu + Z \cdot \sigma = 5 + (.31)(10) = 8.1$$

Exercise #4

A lecturer commutes daily from his suburban home to his midtown university. The average time for a one-way trip is 24 minutes, with a standard deviation of 3.8 minutes. Assume that the distribution of trip times to be normally distributed.

a)What is the probability that a trip will take at least ½ hours?

b)If the office open at 9.00 A.M and the lecturer leaves his house at 8.45 A.M daily, what percentage of the time is he late for work?

c)If he leaves the house at 8.35 A.M and coffee is served at the office from 8.50 A.M until 9.00 A.M. what is the probability that he missed coffee?

d)Due to unforeseen incident, the journey takes longer than usual. If 15% is considered as unusual journey time, what is the length of unusual journey time.

e)Using Binomial distribution, find the probability that 2 of the next 3 trips will take at least ½ hour.